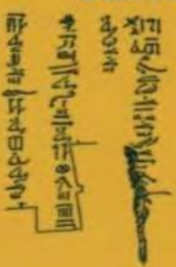
Ancient Egyptian Science
A Source Book



By
Marshall Clagett

VOLUME THREE
ANCIENT EGYPTIAN MATHEMATICS

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Jacket illustrations: *Front top*: The beginning of the first three columns of the hieratic text of the Rhind Mathematical Papyrus (cf. below, page 326, Fig. IV.2a, Plate 1). *Front bottom*: Signs for Egyptian Horus-eye fractions (cf. below, page 379, Fig. IV.3). *Back*: Hieratic fragments from Berlin Papyrus 6619 (cf. below, page 416, Fig. IV.13).

End paper illustrations: *Front*: The hieratic texts of Problems 51 and 52 of the Rhind Mathematical Papyrus (cf. below, page 362, Fig. IV, 2kk). *Back*: Problems 59 and 59B and Problem 60 from the same text (cf. below, page 366, Fig. IV.200).

To my long-time friend and colleague

Herman Goldstine

Preface

It was my hope, expressed in the prefaces of Volumes One and Two, to complete this source book on Egyptian science in a third volume, and, I now add, to do so by the end of the twentieth century. However, it became apparent to me as I approached the end of my treatment of mathematics, that the additional sections on Egyptian medicine and biology as well as on ancient Egyptian representations of nature promised for this volume would produce an unwieldy tome. So I took seriously the declaration of Falstaff in *King Henry the Fourth Part I* (Act V, Scene iv) that the "better part of valor is discretion." The result is that the full treatment of mathematics will see the light somewhat earlier than I expected. Needless to say, I do intend and hope to complete a final, fourth volume.

The organization of the subjects of this volume follows the pattern used in the preceding ones. Part I (Chapter Four–the chapter numbers for the whole work are successive from Chapter One in the first volume) is a long analytical and discursive section on the nature and the procedures of Egyptian mathematics. It is divided topically. Part II comprises a series of the six most important mathematical works translated into English from their hieratic texts. Part III includes a bibliography, an index of Egyptian words, and a second index of proper names and subjects, while Part IV is a very full collection of illustrations (145 pages). These will be useful to anyone following the expositions in Chapter Four and in the introductions to and notes for the documents. Among the illustrations are included copies of the hieratic texts and hieroglyphic transcriptions of the documents for the convenience of a reader who wishes to consult the original hieratic texts.

As in the earlier volumes, I have attempted here not only to give a discourse on the nature and accomplishments of Egyptian mathematics but also to inform the reader as to how our knowledge of Egyptian mathematics has grown since the publication of the first editions of the Rhind Mathematical Papyrus toward the end of the

19th century. Thus I have liberally quoted and discussed throughout the volume the interpretations of such authors as Eisenlohr, Griffith, Hultsch, Peet, Struve, Neugebauer, Chace, Glanville, van der Waerden, Bruins, Gillings, and others. I also consider some of the studies of more recent authors such as those of Couchoud, Caveing and Guillemot.

I must single out for specific praise Chace's splendid edition of the *Rhind Mathematical Papyrus*, which supersedes all prior editions. Accordingly I thought it useful to copy Chace's plates of the text of that papyrus (see Figs. IV.2a-IV.2aaa below). Students of the ppayrus over the last seventy years owe much to the American Mathematical Association for publishing Chace's work I also found beneficial for the completion of this volume the sections H, I, J, and K of W. Kelly Simpson's edition of Reisner Papyrus I, presenting, as they do, account lists with calculations that reveal an interesting contrast to the other documents, which appear to be primarily handbooks of model problems. Therefore, I thank the Museum of Fine Arts of Boston for permitting me to copy the plates of those sections (see Figs. IV.18a-j).

As before, I acknowledge a special debt to Otto Neugebauer for his help and encouragement, though he died just after the completion of Volume One and thus well before I had completed Volume Two and had given much thought to how to organize Volume Three and what to include in it. But, as will be seen by the reader, Neugebauer's keen works on Egyptian mathematics have often been cited below. I also wish to say how useful R.J. Gillings' trenchant treatment of Egyptian mathematics has been and how much I have admired his efforts to reconstruct the fundamental concepts and practices of that mathematics. Though he often admits the lack of direct textual evidence for some of his imaginative interpretations, he usually keeps within the bounds of well-attested Egyptian techniques and perceptions. The reader will discover, however, that I do not always agree with Gillings' conclusions.

It will be readily evident to any one who has studied the first two volumes that I have made some changes in the computer programs used in my preparation of camera-ready copy for Volume Three. The chief change has been the substitution of Hans van den Berg's hieroglyphic program "Glyph for Windows" (prepared at the Centre for Computer-aided Egyptological Research at Utrecht University) for my own way of creating glyphs with the now defunct Fontrix program mentioned in the preface to Volume One. The Dutch program has the great technical advantage of being a vector system of hieroglyphic composition which retains the integrity of the glyph regardless of the glyph's size, while the simple bitmapping system of Fontrix used in the prior volumes tends to reveal the dotted-line composition of the glyphs when they are enlarged. The new system has been designed to work with Word for Windows and functions best with that word processor and its True Fonts. Hence I have abandoned not only Fontrix but its correlative printing program Printrix. However, I confess to missing the ease with which new glyphs, or in fact any kind of new fonts, can be created with Fontrix.

Once more I must thank my secretary Ann Tobias, who retired as the indexes of this volume were being completed. Indeed she continued until the very day of retirement to render all of the same kinds of expert aid which I have so gratefully lauded in the prefaces of the earlier volumes. The final proofreading was taken up ably by her successor Judy Wilson-Smith.

Finally I must thank the American Philosophical Society for its long support of my work, and within that organization I owe a special debt, first to Herman Goldstine, its recently retired Executive Officer, to whom I have dedicated this volume; then to Carole LeFaivre-Rochester, the Society's Editor, who has so consistently and imaginatively helped in the publication of these volumes on ancient Egyptian science; and lastly to Susan Babbitt, the skillful copy-editor of this and the preceding volume.

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CHAPTER FOUR

Ancient Egyptian Mathematics

Quantification in the Early Dynastic Period and the Old Kingdom

Before describing the specific features of the mathematics of the ancient Egyptians, I wish to stress the overall aspects of quantification which developed in the early Pharaonic period. The most primitive uses of mathematics resided in counting, say the inventories of possessions, products, prisoners, or the like, of which there are examples discovered in the earliest tombs at the very beginning of or shortly before dynastic times, and I have discussed these examples in Volume One of this work. Small and mid-scale measurements of length and ultimately of land areas and volumes of solid materials were developed out of the lengths or widths of parts of the body, such as cubit arm lengths, widths of palms, hand spans, and fingers. Liquid and dry grain measures came from the use of common vessels, as cups and jugs for grain and beer, and silos and sacks for larger amounts of grain. The counting of paces or crudely timed boat movements could have lead to the development of the linear measuring of longer distances. The counting of repetitious natural phenomena like the darkness of night and the light of day (and their variations in lengths), lunar, solar, and celestial risings, culminations, and settings led to the establishment of the conventional measures of time like months, years, days, and hours, as I have shown in considerable detail in Volume Two. The counting of the successive years during which kings ruled led to a convenient measure of long term time periods, which has also been pointed out in connection with my presentation of the annals written on stone that characterize the successive reigns of the pharaohs from the beginnings of the united Egyptian kingdom in about 3000 B.C. All of these efforts to measure were facilitated by practical inventions.

The refinement of linear measurement in surveying and building led to the invention of scaled rules (i.e., cubit-rods or double cubit-rods) for shorter measures and rope lengths for longer measures to serve builders and land surveyors.[1] Indeed, it is probable, as Herodotus pointed out in a rather fanciful and anachronistic account of the activity of the Middle-Kingdom pharaoh Sesostris, that at least a stimulus to the development of land measurement (practical geometry) among the Egyptians was the need to measure the land (or, remeasure the land after its flooding by the Nile):[2]

> It was this king (i.e., Sesostris), moreover, who divided the land into lots and gave everyone a square piece of equal size, from the produce of which he exacted an annual tax. Any man whose holding was damaged by the encroachment of the river would go and declare his loss before the king, who would send inspectors to measure the extent of the loss, in order that he might pay in future a fair proportion of the tax at which his property had been assessed. Perhaps this was the way in which geometry (i.e., land measurement) was invented, and passed afterwards into Greece....

In the preceding volumes I have noted some of the uses of mathematics by the ancient Egyptians. For example, we saw in Volume One that the Egyptians used a different glyph for each separate power of ten when counting items in the hieroglyphic form of writing: 𓏺 =1, 𓎆=10, 𓍢=100, 𓆼=1000, 𓂭=10,000, 𓆐 =100,000, and 𓁨=1,000,000 (see Fig. I.10 in Volume One and also the glyphs in the leftmost column in each of the tables of Figs. IV.1a and IV.1b below; various forms of their hieratic counterparts are found in the other columns of these same figures). Not only do these numbers appear in tomb inscriptions from the earliest dynasties, but even on the very early mace-head of the probably predynastic King Narmer, where, as I noted in Vol. 1, p. 6 (see Fig. I.9), "in the lowest register are recorded 400,000 oxen, 1,422,000 goats, and 120,000 prisoners, no doubt the fruit of Narmer's victory" over the northern Egyptians.

These hieroglyphic numbers were also plentiful in the *Early Egyptian Annals on Stone*, which are customarily called the Palermo Stone from one of the main fragments of that document (Document I.1). From perusing that document we find first of all how crucial counting was in the administration of the kingdom for the biennial survey of the wealth of the kingdom, specified as the "counting" (*tnwt*), and how that system of counting the wealth developed into a system of dating by regnal years; see Vol. 1, pp. 50-53. We also see from the very beginning of that document, in the reign of Aha in the first dynasty, that the years, the days, and the months have been numbered: "Year [X +7 (?)] [The last civil year of the reign of the King, of which he reigned the first] six months and seven days" (Vol. I, p. 68).

In the *Annals*, in the section devoted to the reign of the second monarch (Djer ?), we note for the first time the practice of recording, for each year, the greatest height reached in the annual rising of the Nile: "[Nile Height]: 6 cubits, 1 palm" (*ibid.*, p. 69). In a later reigns we find heights as finely measured as "4 cubits, 2 palms, 2 2/3 fingers" and "2 cub., 3 pal., 2 3/4 fing." (*ibid.*, pp. 80-81). Indeed, this is the earliest source I know for the use of fractional measures, the fractions being the palm (=1/7 of a cubit) and the finger (=1/4 of a palm and thus =1/28 of a cubit) and even further fractions of these fractional parts, where we find special signs for 1/2, 2/3, and 3/4; see Fig. IV.0. But it was only somewhat later that we find the whole spectrum of numerically expressed fractions as unit parts: 1/2, 1/3, 1/4..... For early representations of the unit fractions in both hieroglyphics and hieratic writing, see Fig. IV.1c, Table CC. Only 2/3 and 3/4 appeared with numerators greater than one, as we might say today. (We shall have a great deal to say later about the skill of the ancient Egyptian mathematicians in manipulating with these unit fractions.) I discussed the Nile measurements at some length in the first volume (*ibid.*, pp. 109-113). In the second volume (see the Index under "Nile risings") I stressed the possible significance of Nile risings for the establishment of a seasonal calendar of 365 days, which developed into the well-known Egyptian Civil Year that was used without refinement during the whole Pharaonic period up to the accession of the Ptolemaic kings in Egypt.

Another important use of numbers in the *Early Annals on Stone* was to describe the quantity of land involved in various gifts to the temples. In describing this usage, I have already noted that a kind of primitive decimal place-value system was sometimes used (see Volume One, pp. 56-57):

> The gifts of land mentioned in the *Annals* also yield precious information to the historian of Egyptian numeration and measurement. The measures of land in these grants involve *ḫꜣ* (a 10-aroura measure), *stꜣt* (unit-aroura measure), *rmn* (1/2 aroura), *ḥsb* (1/4 aroura), *sꜣ* (1/8 aroura), *mḥ* (a cubit-area, i.e., 1/100 aroura). If we start with a "cubit-area" [or *tꜣ*, so called from its meaning as "land-strip",] (i.e., an area 1 cubit wide and 100 cubits long), then 1/8 aroura = 12 1/2 cubit-areas, 1/4 aroura = 25 cubit-areas, 1/2 aroura = 50 cubit-areas, 1 aroura = 100 cubit-areas, and 10 arouras = 1000 cubit-areas (which reveals why... [𓆼] read as *ḫꜣ*, meaning 1000, was used for the 10-aroura measure; see Fig. I.50 [in Volume One and the Griffith and Gardiner citations in note 14 below commenting on the different interpretations of some of these measures by Helck]).
>
> In expressing the various areas of land grants, the annalist has sometimes made use of a primitive place-value system... In this system we find numbers where the 10-aroura measures are numbered first in units without an expected preceding 10-aroura sign, then followed by an ... aroura sign which is itself followed by the counting of those unit-arouras in units. We also find numbers where 100-aroura measures are counted in units without any preceding 100-unit sign, then followed by the 10-aroura sign which is itself followed by the counting of 10-aroura measures in units. But the annalist was not always consistent in the use of the place-value technique...and in fact the use of the technique in later times was rare.

There is no reason to believe that this primitive system was of any special use in simplifying calculations involving addition, subtraction, and multiplication, as was the case later when Indo-

Arabic numerals with their carefully delineated decimal columns came into use. Hence without any special calculating benefit evident in the early Egyptian system, there was no particular stimulus to develop it further when the abbreviated hieratic writing of numerals became widespread in the Middle Kingdom, and the early place value system remained an historical oddity of the hieroglyphic writing of numerals.

It should also be noted that the *Early Annals on Stone* contained several references to surveying activity for the laying out of the ground plan or "stretching the cord" as the first step in the construction of temples or mansions (see Vol. 1, p. 50 for references to stretching the cord in the reigns of King Den in the first dynasty, King Ninetjer in the second, and King Djoser in the third; for further discussion of stretching the cord, see *ibid.*, pp. 124-26). Incidentally, Fig. IV.23 illustrates a pair of farm hands carrying a cord to be used for measuring a field. We have also seen in Volume I (pp. 145-46) that land transfers were mentioned in the tomb of Metjen, the administrative head of the Provision-Bureau at the end of the third dynasty and the beginning of the fourth. These imply, as I said there, "that full play was given to measurement, and particularly to land measurement."

These examples we have given of the use of numbers and some measures in the early periods do not of course tell us much about the date of the origin of the procedures of Egyptian mathematics, but they allow us not to be so pessimistic about the probability that a good many arithmetic and geometrical rules were in place from the beginning of the dynastic period. Even so, we can understand the somewhat negative assessment by Eric Peet of our lack of knowledge of the origin of Egyptian mathematical techniques:[3]

> Our information on this point is sadly defective. The Rhind Papyrus [our Document IV.1] dates from the Hyksos Period, though it claims to be a copy of a document prepared in the XIIth Dynasty, in the reign of Amenemhet III. This may well be, since both the Moscow Papyrus [our Document IV.2] and the Kahun fragments [our Document

IV.3] date from that Dynasty. But how much earlier must we go to find the beginnings? Surely the complicated fabric of Egyptian mathematics can hardly have been built up in a century or even two, and it is tempting to suppose that the main discoveries of mathematics should be dated to the Old Kingdom. There is a very definite tendency among Egyptologists to put this period down as the Golden Age of Egyptian knowledge and wisdom. There can be little doubt that some of the literary papyri have their roots in this era, as for example the Proverbs of Ptahhotep, and the antiquated constructions of the medical papyri make it possible that the science of medicine, such as it was, had its spring in the Old Kingdom.

Of definite evidence for this early date there is none. All we know is that by the beginning of the First Dynasty the system of notation was complete up to the sign for 1,000,000.... In the [end of the IIIrd and beginning of the] IVth Dynasty we find in the tomb of Methen [Metjen] that the land measures of the Rhind Papyrus are already in full development in a form which involves correct determination of the area of the rectangle, but not of necessity of the triangle or circle. There appears to be no early evidence with regard to measures of capacity, though one may almost take it for granted that with the measurement of the field on which the corn was grown went that of the containers in which it was stored and sold. That measurement by weighing was practised can hardly be denied in view of various objects of Old Kingdom date which can scarcely be anything but weights, as for example the stone weight of Khufu, formerly in the Hilton Price collection, though the attempts to establish a standard from these objects have been far from satisfactory.

From these feeble indications we pass straight to the fully developed mathematical system of the XIIth Dynasty, the early stages in the building up of which are entirely concealed from us.

Peet's assumption about Egyptian knowledge of the area of a rectangle in early times is surely correct in view of use of the measures of cubit-areas and arouras and their multiples and fractions in both Document I.1 and I.II in Volume One; see, e.g., Fig. I.50. But he is also correct in noting lack of evidence of mathematical tracts (with sample problems and tables) until the Middle Kingdom. And so we shall discuss shortly the cautious steps taken from the practical measurements mentioned above toward some simple general conceptions of numerical relationships or formulas to serve as models for budding calculators and even toward the concept of proof or at least of the testing of the accuracy of a calculation. But before we do this, we should discuss the various measures employed by the early Egyptians, which for the most part are found in the mathematical documents that follow this chapter.

Egyptian Measures[4]

At this point I refer the reader to the lists of ancient Egyptian measures embraced in Figs. IV.1c, IV.1g, IV.1h, IV.1j, and single out here the principal ones that figure in the mathematical documents below. As indicated above, the most common of linear measures was the royal cubit *(meh nesut* or, probably in more exact transcription of consonants alone, *mḥ nỉ-swt)*. It was approximately 20.6 inches (i.e., 52.3 cm.),[5] dividable, as has been mentioned above, into 7 palms or 28 fingerbreadths (also commonly called "fingers" or "digits"). The cubit was used for smaller lengths, while the *khet* (or more fully *ḫt-n-nwḥ*), a measure of 100 cubits, was used for field measurements, and the *ater* (or *ỉtr)*, the "river measure," equal to 20,000 cubits (i.e., about 10.5 km.) was the longest measure used for larger fields and for itinerary purposes (e.g., see my Vol. One, pp. 492-94, 507 n. 1) and thus was similar to the Greek *schoenus*. The Egyptian cubit-rod was first investigated in detail by Lepsius (see note 4) and I have given his reproductions of various wood, stone, slate, basalt, bronze, and talc cubit- or ell-rods as my Figs. IV.24 (Tafeln 1-5). He also gave us in tabular form the significant facts concerning the division of these rods and I have slightly emended the transliterations; my additions are given in brackets:[6]

meh nisut *[mḥ nỉ-swt]* "royal cubit" = 7 palms

= 28 digits.

meh nedjes *[i.e., mḥ nḏs]* "short cubit" = 6 palms

= 24 digits.

remen [i.e., rmn] "upper arm" = 5 palms = 20 digits.

djeser [i.e., ḏsr] ["the bent arm"] = 4 palms = 16 digits.

pt [or *pḏ* or *shat*] *aa [i.e.,šꜣt ꜥꜣ]* "great span"

= 3 1/2 palms = 14 digits.

pt [or *pḏ* or *shat*] *nedjes [i.e.,šꜣt nḏs]* "small span"

= 3 palms = 12 digits.

[with back of hands bent down] = 2 palms = 8 digits.

[fist, *ḫfꜥ*] = 1 1/2 palms = 6 digits.

"handsbreadth" = 1 1/4 palms = 5 digits.

, , *[she]sep* [i.e., *šsp*] = 1 palm = 4 digits.

, *djba*, digit (subdivided to 1/2, 1/3, etc. to 1/16) [and to be distinguished from the finger used as 10,000]

= 1/4 palm = 1 digit.

Also of interest is Susan K. Doll's description of the cubit-rod of the Chief of Treasury Maya of Dynasty 18.[7] It is at the Louvre Museum, N. 1538 (see Fig. IV.25), and it is the cubit-rod more clearly reproduced in the drawing by Lepsius in my Fig. IV.24 (Tafel 2) (a):

> Length 52.3 cm., width 3.2 cm....
> The wooden rod measuring the length of the royal Egyptian cubit is painted black and has white markings. The rod is rectangular in section, with a beveled edge between the top surface and the marked side.
> The royal Egyptian cubit was seven palms long, each palm being divided into four digits or fingers, and each digit further subdivided. These subdivisions up to a sixteenth of a digit are marked by strokes on one of the vertical sides. On the beveled edge are two registers, one showing the subdivision of the digit written out as fractions, and the second containing markings for other kinds of internal cubit divisions, such as that for the short cubit (six palms), the great and little *shat*, and various other measurements based on the hand. Along the flat top surface are the names of the gods who preside over each digit. On the bottom of the rod and on its back are three columns of dedicatory inscriptions.

She also mentions the existence of another cubit-rod belonging to Maya, which is now in Turin. This is the cubit-rod reproduced by Lepsius and given in my Fig. IV.24 (Tafel 1) (b). The fact has often been noted that the *remen*, listed in Lepsius' table as the initial division of the cubit after the long and the short cubits and as being equal to 5 palms, is approximately equal to half the diagonal of the square with a side of one cubit (i.e., 7 palms).[8] From this empirical fact, there have been exaggerated claims that the Egyptians had knowledge of the Pythagorean theorem, which is, of course, a formal Euclidean theorem of the *Elements* (Prop. I.44), expressed as part of a logical structure with definitions, postulates, and axioms never realized or specified by the Egyptian geometers. But in fact the inclusion of the *remen*, like the inclusions of the other linear measures, was a measuring convenience that allowed for the laying out of square areas and their fractions.

In his comments on Lepsius' analysis of cubit-rods, Griffith notes that the evidence derived from that analysis about the cubit and its division must be received with caution, "for they are often very carelessly inscribed."[9]

As a further observation on the cubit-rods, I call attention to the remark of Ms. Doll's that the Parisian rod contained the names of the gods presiding over each digit. This is just another indication of the religious connections with science in our account in these volumes. Indeed we should point out the fact that in addition to the cubit-rods (mainly of wood) by which the actual practical measures were made, there is a class of ceremonial cubit-rods usually of stone and often kept in temples and the tombs of high officials. Hayes describes two such rods in the collection of the Metropolitan Museum in New York (see Fig. IV.26):[10]

> Among the units of measure with which the Egyptian scribe was required to be familiar one of the most important was the royal cubit of seven palms (20 9/16 inches).... Our own knowledge of this standard unit of measure [in the Metropolitan] stems not from the relatively simple wooden cubit rods actually used to record and lay out measurements, but from the elaborately inscribed ceremonial cubit rods of stone, some of which seem to have been kept in the temples of the gods while others were buried in the tombs of royal architects and other prominent officials of the New Kingdom. Two such cubit rods, carved, respectively, of chert and green slate, are each represented in our collection by a single fragment (see fig. 263 [*Author:* my Fig. IV.26]) comprising less than one-sixth of the original rod but sufficing to show the character of the monument and the nature of its texts. Both rods belonged to a type which carried, in addition to all the duly labeled divisions and subdivisions of the cubit, "such a bewildering amount of assorted information that 'almanac' seems a better word than 'cubit rod' by which to describe them."[11] The fragment illustrated, for

> example, preserves on the front edge of the rod the first and second "digits" of the cubit subdivided, respectively, into two halves and three thirds, on the bevel the names of the corresponding nomes of Upper Egypt (Elephantine and Edfu), and on the top of the rod the names of the deities associated with the individual digits (in this case Re^{c} and Shu). On the bottom of the rod the first line of text starts off with the words "The hour according to the cubit: a jar (?) of copper filled with water...." which have been thought to refer to a water clock and the manner of reading it. The second line, the mid-section of which probably contained the names and title of a king, begins: "This is a communication for those who shall be introduced [into Mendes]...." The third line would seem to have given the relative heights of the annual inundation at different places along the Nile Valley. On the back of the rod the list of nomes is resumed with the names of the Seventh and Eighth Nomes of Lower Egypt (Metelis and Pithom). The little text on the end of the rod tells us that "The cubit is life, property, and health, the repeller of the rebel, the...going forth of Chnum [or Khnum], who is Es[neh] (?)." The piece is of unknown provenience.... The [second rod, i.e., the] fragment of the slate rod...comprises the third to seventh digits of the cubit and bears additional portions of the texts already referred to, including, in the first line on the bottom, "measurements in cubits and palms according to the months of the year," "possibly a table by which readings of a sun-dial [or, better, a shadow clock?] might be interpreted."[12]

Similar ceremonial or votive cubit-rods, or "Weihellen," as Ludwig Borchardt calls them, from the Cairo and Berlin Museums were noted to contain some information and tables that might also pertain to water- and shadow-clocks. They were long since discussed (but without claiming certainty) by that great student of time measurement and are shown in my Fig. IV.27a,[13] and it is evident that his discussion of the possible uses of the sacred cubit-rods in determining the hours influenced the more recent discussion of the use of such rods as hour clocks.

When we proceed from linear measures to those of areas, we remind the reader of what we have said about land measurements present in the documents of the Old Kingdom, like the inscriptions in the tomb of Metjen and in the *Annals on Stone*. The area measurements from the Old Kingdom have been summarized by Helck [with the author's bracketed comments]:[14]

> 100 square cubits = 1 *t3* = 10 x 10 cubits = 27.565 sq. m.
> 1000 sq. cubits = 1 *ḫ3* = 10 x 100 cubits =275.65 sq. m. [see note 14 for value of the *ḫ3* as 10 arouras].
> 10000 sq. cub. = 1 *sṯ3t* [or 1 "aroura," a Greek term often used to translate *sṯ3t*] =100 x 100 cub. = 2765.5 sq. m.
> The following fractional divisions of the *t3* are found in the Old King.: 1 *rmn* = 1/2 *t3*;1 *ḥsb* = 1/2 *rmn*; 1 *z3* = 1/2 *ḥsb*.
> [See note 14 for fractions of the *sṯ3t* found in the *Annals on Stone* instead of those of the *t3* given in Helck's table.]

Examples of these area measures as used in the Middle Kingdom, and slightly later, can be found in Problems 48-55 in Document IV.1 and Problems 4, 6-7, 11, and 18 in Document IV.2 below. I shall discuss the geometrical formulas involved in these problems later when I discuss the character of Egyptian geometry, but it will be useful to note the following conclusions of A.B. Chace regarding the area problems of Document IV.1, with additions of my own in brackets:[15]

> The only problems dealing with area are 48-55. The units of measure in these problems beside the cubit...are, first, the linear unit called *khet* which is 100 royal cubits, and, second, the square *khet* called *setat* [i.e., *setjat*, see the table given above], which is 10,000 square cubits. The area of a field is expressed in terms of the *setat* and fractions of a *setat* in much the same way as the measure of a quantity of grain is expressed in terms of the *hekat* [which in this vol-

ume I have transcribed as *heqat*] and fractions of a *hekat* [see note 13 above and my discussion of volumetric measures in the paragraphs below]. In the first place, the Egyptians used the fractions 1/2, 1/4, and 1/8 of a *setat* in the same way as they used the "Horus eye" fractions [of a *hekat*], and for these fractions also they had special forms [differing from those of the Horus eye fractions, but which like them are expressed here in bold type]. Then smaller portions of a *setat* were expressed in terms of a unit that they called "cubit" [or "cubit-of-land"] and seemed to have thought of as a strip 1 *khet* or 100 cubits long and 1 cubit wide. I shall call this unit a "cubit-strip" [and this name is adopted also in this volume]. 100 cubit-strips make a *setat*, and as **1/8** of a *setat* is equal to 12 1/2 cubit-strips, calculations with this system of units are not quite as simple as with the *hekat* system, where the smallest "Horus eye" fraction is equal to a whole number of *ro* [see the next paragraphs below]. Thus in Problem 54 we have 1/5 of a *setat*, which is 20 cubit-strips, expressed as **1/8** *setat* 7 1/2 cubit-strips. The double of this is **1/4 1/8** *setat* 2 1/2 cubit-strips, and so on....

Going on to volumetric measures, we can note two systems, often intertwined in problems related to the size of the containing space in cubic-cubits and the amount of the contained grain or liquid in *hen* or *heqat* and sometimes independently as in the case of the volume of an architectural building or element, like a chapel, where the cubic-cubit and its fractions appear by themselves. The mixed system is found, e.g., in the granary problems (41-46) in Document IV.1. The system using cubic-cubits and their fractions alone is found in Document IV.6, where the volume of space cleared of rubble is used to determine the number of man-days of workers. In discussing that document below, I have noted that the ultimate volume produced by the product of linear cubit measures is understood to be in cubic-cubits. Furthermore, the fractions of those cubic-cubits are expressed often by the linear names of "palms" and "fingers." In such cases those terms stand for 1/7th and

1/28th of a cubic-cubit. In the system involving heqat, we note that in the Middle Kingdom the largest measure was the "sack" or "khar" (*ḫꜣr*) = 20 "heqat" (*ḥḳꜣt*), [i.e., 5 4-heqat] = 2/3 of a cubic-cubit (see Doc. IV.1, Problems 41-46). The heqat in turn = 10 hin (*hnw*), (*ibid.*, Problem 80). The hin = about .48 liter.[16] Hence, in the Middle Kingdom at least, the heqat = about 4.8 lit. and thus the sack = about 96 lit. With his eye on the Rhind Papyrus, i.e., our Document IV.1, its editor, A.B. Chace makes the following observations on measures of capacity, with my additions in brackets:[17]

> The unit of volume or capacity, used especially in measuring grain, was the *hekat* [commonly transcribed in my volumes as *heqat*], which can be determined as 292.24 cubic inches, or a little more than half a peck. This was divided into 320 parts called *ro* [or *re*, 𓂋, with the number of parts, i.e., the denominator of the fraction, indicated in regular numerical glyphs by strokes below the "part" glyph], but the Egyptians also used as fractions of a *hekat* the fractions whose denominators are powers of 2 down to 1/64, 1/64 of a *hekat* being 5 *ro*. This series of fractions was peculiarly adapted to multiplication by doubling or halving. They were written in a special notation and have been called "Horus eye" fractions....[see note 13 above and its references to Horus eye fractions].
>
> Beside using the "Horus eye" notation for parts of a *hekat*, the Egyptians had special hieratic signs for the numbers from 5 to 10 when used to express *hekat*. These signs seem to be ligatures of dots, the sign for 10 being a long vertical stroke, representing perhaps ten dots [or little circles] one above another [and they had a peculiar sign used alone and with other signs for different kinds of grain and with expressions for a large quantity of grain, the basic sign  being a corn-measure lying on its side with grains pouring out of it].... When the amount was equal to or more than 100 *hekat*, this sign was written with the number of hundreds before it, and the signs for any smaller number of

> *hekat* next after it. Also 50 *hekat* and 25 *hekat* were put down as 1/2 and 1/4. The number of whole *hekat* was followed by "Horus eye" fractions and by *ro* and fractions of a *ro* [for example, see Problems 47, 68-70 in Document IV.1]. In the case of 2, 3, or 4 *ro* the sign 𓂋 for the word *ro* was written under the number, while this sign without a number stood for 1 *ro*, and the fractions of a *ro* came after the sign.
>
> Furthermore the Egyptians had not only the system of a simple *hekat* and its parts and multiples, but also systems of a double *hekat* and a quadruple *hekat* with their parts and multiples, each part or multiple of a double *hekat* being twice the corresponding part or multiple of a simple *hekat*, and each part or multiple of a quadruple *hekat* [being] four times the corresponding part or multiple of a simple *hekat*....

Though the word *hen* or *hin* is used for the measure of liquids generally and of grain as well, there are specific words used for each liquid, but I note here only the common word for a measure of beer, namely "jug" ([hieroglyphs], *ds;* e.g., see Document IV.1, Problem 71 and Document IV.2, Problem 9, Col. XVII, line 3). Other specific measures will be noted and discussed in the documents below and have been conveniently listed by Helck.[18]

Systematic Mathematical Treatises and Tabular Aids

I have given some examples of the early uses of quantification in the form of counting and land measure in the first section of this chapter and listed the standard Egyptian measures in the second. It will be evident from the rest of the chapter that the dominant aspect of ancient Egyptian mathematics continued long after the first period to be the usage of practical techniques governed by the need to measure and to count. But by the time of the Middle Kingdom, numerous tracts were written in hieratic on papyrus (or in one case leather) with model problems that served to instruct the accountant, surveyor, tax assessor, builder, baker, brewer, and other computists, making their tasks simpler. Also numerous tables were composed to speed up the necessary routines of calculation. It will be evident that the six documents that I have included at the end of this chapter allow us to delineate clearly the principal characteristics of ancient Egyptian mathematics and its aids. Though we do not have enough sure data to date all of the documents precisely, even those whose extant copies were made later than the Middle Kingdom seemed to depend on Middle Kingdom sources whether for the whole tract, as is certain of Document IV.1, or for the forms or techniques of solution of particular problems, as is probably the case of Documents IV.2-IV.5. Finally, Document IV.6 is essentially an accounting document involving actual volumetric determinations according to simple formulas rather than a collection of model problems. Hence more approximations and difficult fractional multiplications are present. Before considering the essential features of Egyptian mathematics, I should like to list the six documents, briefly identifying them and giving their probable dates.

1. Document IV.1: the Rhind Mathematical Papyrus, written down in the 33rd year of the Hyksos King Apophis (ca. 1585-1542 B.C.); but, as we are told by the scribe, it was copied from an earlier version written during the reign of the 12th dynasty King Amenemhet III (ca. 1844-1797 B.C.). We owe this early date to the introductory paragraph of the tract, as the reader can readily see. As I shall show at some length in this chapter, and in the translation of the document, it is not only the longest mathematical document, but it has the most varied content and tells us more about Egyptian mathematics as a whole than any other document. The first section contains the so-called "Table of Two," which, in modern terms, we could regard as a table for reducing fractions with numerator 2 and denominators the odd numbers from 3 to 101 to sums of unit fractions, i.e., fractions with numerator 1, but which is more accurately presented in Egyptian terms as a series of divisions of 2 by the odd numbers from 3 to 101 such that the quotients are the sums of fractions each of which has the numerator 1. This document also includes a great many different kinds of problems

(see my list in the introduction to the document). The problems illustrate amply the Egyptian techniques of doubling, halving, taking 2/3, multiplying by 10, multiplying fractions by each other, finding unknowns, computing the areas and volumes of figures, calculating the slopes of triangles, and so on, as we shall demonstrate shortly.

2. Document IV.2: the Moscow Mathematical Papyrus. This copy appears to have been written down in the 13th dynasty (from about 1783 B.C. to after 1640 B.C.). But its editor believed it to be dependent on a work of the 12th dynasty (see the introduction to Document IV.2, n. 4), perhaps dating from about the same time as the earlier copy from which the Rhind Papyrus was copied. It contains examples (chaotically arranged) of many of the same types of problems as the Rhind Papyrus. Its chief interest lies in its geometrical problems, such as its correct solution of the volume of the frustum of a square pyramid (Problem 14) and one that may give the surface area of a hemisphere (though this is much disputed, as I note below). It also contains a number of beer and grain problems which are difficult to understand, as the notes to the translation of the document reveal; but they are not very crucial for our knowledge of Egyptian mathematics.

3. Document IV.3: the Kahun Mathematical Papyrus, which is believed to have been composed in the second half of the 12th dynasty. There is nothing very original in the fragments making up this document. It contains a fragment of the Table of Two, and an interesting problem involving numbers in arithmetic progression. For further comments on its contents, see the introduction to the document below.

4. Document IV.4: the Berlin Papyrus 6619. The fragments contained in this document date from about the same period as the two preceding documents, namely some time from the second half of the 12th dynasty through the 13th, as I remark in my introduction to the document. The document consists of two arithmetic problems which are similar to finding by false position two unknowns when given two simultaneous equations.

5. Document IV.5: the Mathematical Leather Roll of the British Museum. The authors of some laboratory notes on this leather roll suggest its date as 17th century B.C. (see the Introduc-

tion to Document IV.5, note 3). It consists of duplicate copies of 26 sums of unit fractions, that is, of a series of equalities that were no doubt aids for calculating with fractions. I shall discuss this document later in this chapter.

6. Document IV.6: sections G-I of Reisner Papyrus I of the Museum of Fine Arts in Boston. Its editor, W. Kelly Simpson, believes these sections date between IV Peret 6 and II Shemu 20 of Sesostris I in the 12th dynasty (see the Introduction to Document IV.6, n. 3 and the text which that note supports).

As I shall explain in the translation of the document, it is an actual account record of the volumes of material removed in a building project and the man-days required in the undertaking. Thus, as an accounting document, it contrasts sharply with the preceding documents that present model problems of calculation and /or tables to assist in their preparation.

Basic Concepts of Egyptian Mathematics

Many students of Egyptian mathematics have singled out the fundamental importance to its whole structure of the concepts of doubling a quantity, halving it, taking two-thirds of it, multiplying it by 10, and taking 1/10 of it, and of finding fractional multipliers by the use of reciprocals of the products resulting from the use of the above-mentioned multipliers. Singly, or in conjunction with these concepts, along with the ordinary addition of numbers (both whole numbers and fractions, the latter with the concept of "auxiliary numbers or red auxiliaries" not unlike the assumption of common denominators in later arithmetic), are those used throughout the documents whose translations appear below, as I shall illustrate shortly. But there are other fundamental concepts that emerge from an examination of the documents. The following stand out:

(1) The fundamental concept of counting as well as that of measuring. These we have already mentioned in the preceding sections, and with them the fundamental signs or characters invented to keep track of the results of the counting and measuring. We should also point out that connected with the early concept of mathematics as measurement is the concept of "rounding off" or

approximating fractions when it is obvious that the efforts of multiplication and division of fractions produce unit fractions so small (i.e., with denominators so large) that there is no practical way of measuring them. This approximating of fractions is evident in practical accounting tables like those exhibited in the Reisner Papyri (Document IV.6 below) and in the table of the distribution of portions of bread and beer at the temple of Illahun in the Middle Kingdom (see the end of the section called "Pefsu Problems" in this chapter, text over note 61).

(2) The basic idea of recording the aids to calculation in the form of tables. Examples: (a) the "Table of Two," which (as said before) is a table given in the first part of Document IV.1 for finding the quotients of divisions involving a dividend of 2 and divisors that are successively the odd numbers from 3 to 101, the quotients being expressed as the sums of unit fractions; (b) the "Table of the Division of the first 9 units by 10" that follows after the Table of Two; (c) the "Table of the Division of 100 Heqat by 10 and the succeeding 10 multiples of 10" (Document IV.1, Problem 47); (d) Table for the Multiplication of Fractions (*ibid.*, Problem 61); (e) "Tables of the Reckoning of Henu from Horus-Eye fractions of a Heqat" (*ibid.*, Problems 80-81); and (f) a Table of Equalities of Fractions (Document IV.5). All of these tables will be discussed in later sections of this chapter.

(3) The concept of model problems and the generalizing of problems seeking an unknown quantity (*aha*) as incipient steps to mathematical generality. This will also be discussed below.

(4) The concept of proof by testing. Throughout Document IV.1 we see the result or solution has been used to satisfy the conditions of the problem, i.e., to say, it is a common practice in the problems to work backwards with the calculated answer in order to show that it fits the enunciation of the problem.

(5) The concept of standard calculating formulas for the determination of areas and volumes in terms of rectilinear dimensions, the figures being squares, rectangles, triangles, circles, trapezoids, rectangular and cylindrical containers. All of this will be discussed in the sections on areas and volumes below.

(6) The concept of the slope of an isosceles triangle (and thus of pyramids and a cone). This will be discussed in the relevant sections on geometry below.

(7) The concept of quadrature, that is, the reduction of curved figures in two or three dimensions to figures bounded by straight lines or planes. This begins a study that leads from Egyptian mathematics through the brilliant efforts of Greek geometry (and above all in the works of Archimedes) to modern geometry, and I shall have something to say of this later in the chapter.

Egyptian Arithmetical Procedures

Modern arithmetical procedures depend fundamentally on the decimal place value system. As I have said earlier, though the mathematicians of Ancient Egypt had a primitive place value system, it never became an essential part of their calculating procedures. However, addition and subtraction depended on shifting to higher or lower powers of ten when the simple operations demanded. But multiplication (and division also, as we shall see later) were accomplished by a series of fundamental multipliers, i.e., multiplying by 1, doublings, taking of 2/3, halvings, multiplying by 10, or taking of 1/10, using as many of these operations as were needed to accomplish the calculation. These procedures are exemplified throughout the solutions of the various problems given in Document IV.1. In the course of solving Problem 32 involving an unknown quantity (an aha-problem) we see the incidental multiplication of 12 x 12 completed as follows:

1	12
2	24
\4	48
\8	96
Total:	144.

In this table we see the first column is headed by the unit multiplier and followed by successive doublings as multipliers, the second column by the multiplicand and the successive products of doubled multipliers, until there are laid out such doubled multipliers that add up to the multiplier specified in the multiplication stated as the objective of the table. The desired multipliers that so add up are checked and the addition of the products given in the second column opposite the checked multipliers gives the final answer, which is indicated below as a total. The same desired multiplication of 12 x 12 could have been performed as in the table I have given in the second version (i.e., my first reconstruction) of Problem 43 in Document IV.1 below, but this time by using one doubling and one multiplication by 10:

1	12
\2	24
\10	120
Total:	144.

And in this reconstruction of Problem 43 we would also see a preceding table yielding the solution of the multiplication of 1 1/3 x 9. Here we find the pivotal role of taking 2/3 of 9, and following it by taking 1/2 of 2/3, i.e., 1/3:

\1	9
2/3	6
\1/3	3
Total:	12.

Finally in Problem 44, we see a table in which occur multiplications by 10, by 20 (i.e., by 2 x 10), by 1/10 (twice), and by 2/3 of 1/10 of 1/10, i.e., by using all of the fundamental multipliers:

1	75
10	750
\20	1500
1/10 [of 1500]	150
1/10 of 1/10	15
2/3 of 1/10 of 1/10	10.

As we saw from the last example, three of the fundamental multipliers used were fractions, i.e., 2/3, 1/2, and 1/10. Before taking up the basic procedures with fractions, we should first remark on the Egyptian techniques of division. If we should look at a table involving the division of 144 by 12, we would see precisely the same two columns of figures as are present in the multiplication of 12 x 12 (except that we have given here the total addition of the first column as well as that of the second):

1	12
\ 2	24
\ 10	120
Total: 12	144.

While the calculations look alike when presented as a table, a question is being asked for division that is different from that for multiplication. Instead of asking for the sum of the products in the second column (24 and 120) that are opposite the checked multipliers of the multiplicand in the first column (i.e., 2 and 10), which add up to the original given multiplier (12), as we were in the multiplication of 12 x 12, we are now asking for the quotient as the sum (12) of the two multipliers (namely, 2 and 10) that together yield a sum of those products in the second column that add up to the dividend (144). As the Egyptians would say, we are "calling up 144 by operating on 12"; thus we are finding the quotient (12) as the sum of those multipliers of the divisor that added together produce the dividend (144).

Let us now shift to a more complicated division involving fractions performed in the course of Problem 66. This division solves the problem of finding out the uniform amount of fat in heqat and ro (= 1/320 of a heqat) distributed each day if we start with 10 heqat of fat for the whole year. After reducing the 10 heqat of fat to ro, i.e., 10 x 320 = 3200, and the year to 365 days, the division of 3200 by 365 (that is getting "3200 by operating on 365"), is accomplished as follows:

1	365 *[cont.]*

2	730
4	1460
[\ 8	2920]
\ 2/3	243 1/3
\ 1/10	36 1/2
\ 1/2190	1/6

Total: 8 2/3 1/10 1/2190.

It is evident from this table that after having produced 2920 by using 8 as the multiplier (obtained by successive doubling), the calculator cannot even add the product of 1 x 365 without exceeding the dividend 3200. Hence he proceeds to the largest allowable fractional multiplier, which is 2/3. But that only gives him 3163 1/3, still short of the dividend 3200. Furthermore, he sees that he cannot use the next largest allowable fraction, i.e., 1/2 as a multiplier of 365, because again he would exceed 3200. So he proceeds to 1/10, the last of the standard fractional multipliers. But now his product is 3199 5/6, still 1/6 short of the dividend 3200. To get the final multiplier of 365 to produce the missing 1/6, he sees that the multipliers 4 and 2, i.e., 6 in total, yield a product of 2190. So he uses the reciprocal of that product, namely, 1/2190 to produce the final 1/6 and the addition of all the checked multipliers will produce the quotient of 3200:365, i.e., 8 2/3 1/10 1/2190. In Problem 66 the reader will note that the answer given in the table is expressed entirely in ro, but that answer is also converted in the preceding enunciation into *1/64* heqat 3 2/3 1/10 1/2190 ro, since 5 ro = *1/64* heqat (that fraction being the last of the 6 Horus-eye fractions of a heqat mentioned earlier). We shall also see in a moment that the use of reciprocals to produce the fractional parts that add up to the final quotient is a very fundamental technique of the Egyptian method of division. Furthermore, this problem with its use of reciprocals as fractional multipliers also reflects one of the principal benefits of the Egyptian procedure of always giving fractional expressions as the sum of unit fractions. Hence we are now ready to discuss in detail the Egyptian procedures with such unit fractions.

Unit Fractions and the Table of Two

Any reader who browses through the six documents that constitute the core of my treatment of Egyptian mathematics will see immediately how prevalent calculations with unit fractions are in the solutions of the problems presented. For example in Document IV.1, by far the longest and most complete document included here, every entry in the Table of Two and all the problems but Nos. 48, 62, and 77-79 involve some calculation with fractions, either as fractions alone or as parts of mixed numbers. I have already noted that, aside from 2/3 and possibly 3/4, the only fractions expressed in the calculations given here are those with numerator 1, to express this in modern terms. As I have pointed out, the use of the number 1 is used here to represent the symbol 𓂋 ("*r*") in hieroglyphics (the mouth-sign meaning "part") which is reduced to a superior dot in hieratic, with the number of the denominator written below these symbols (see Fig. IV.1c, Table CC, cols. Hierog. and Math., or *passim* in Figs. IV.2a-aaa). This practice of writing fractions is followed in hieroglyphics from 1/3 on and in hieratic and demotic from 1/5 on.[19] The signs for the earlier fractions were special and did not follow the general practice for writing fractions, which I have just mentioned. Furthermore, special signs were used for 2/3 and, occasionally, for 3/4, meaning respectively "two of three parts" and "three of four parts." Gardiner's observations on fractions and "part" are worth quoting:[20]

> For the Egyptian the number following the word *r* had ordinal meaning; 𓂋||||| *r-5* means 'part 5', i.e. 'the fifth part' which concludes a row of equal parts together constituting a single set of 5. As being the part which completed the row into one series of the number indicated, the Egyptian *r*-fraction was necessarily a fraction with, as we would say, unity as the numerator. To the Egyptian mind it would have seemed nonsense and self-contradictory to write *r-7 4* or the like for 4/7; in any series of seven, only one part could be the seventh, namely that which occupied the seventh place

> in the row of seven equal parts laid out for inspection. Nor would it have helped matters from the Egyptian point of view to have written  *r-7* (+) *r-7* (+) *r-7* (+) *r-7*, a writing which would have likewise assumed that there could be more than one actual "seventh". Consequently, the Egyptian was reduced to expressing (e.g.) 4/7 by 1/2 (+) 1/14. For more complex fractions even as many as 5 terms, all representing fractions with 1 as the numerator and with increasing denominators, might be needed [or still more terms, e.g., see Document IV.1, Problem 37, where in the proof 10 unit fractions are added to produce 1.]....It is not generally known that the same cumbrous methods of expression were in common use with the Greeks and the Romans.

Gardiner's explanation that 4/7 = 1/7 + 1/7 + 1/7 + 1/7 was not used because it "would...have assumed that there could be more than one actual 'seventh'" does not seem to be the correct explanation for reluctance to write more than 1/7 in the series of unit fractions that one used in the Table of Two and elsewhere to express the quotient as a series of unit fractions in which the denominators differed from each other. The correct explanation seems to me to be that of van der Waerden in his account of the Table of Two given below (text over note 33): "Now if a unit fraction, say 1/7, is doubled, one gets 1/7 + 1/7, and if this is doubled once more, an unwieldy expression like 1/7 + 1/7 + 1/7 + 1/7 is obtained. To avoid this, the Egyptians devised a method to rewrite [it] as a sum of different unit fractions." It is obvious that when the Egyptian mathematician composed tables of equalities giving the sums of unit fractions like those in Document IV.5 (the Mathematical Leather Roll) where, in a sense, numerical quantities alone are given throughout the equality and no specific measurement is involved, unit fractions with the same denominators are often added, as the reader will readily see.

We should realize that in the case of the summing of the 10 unit fractions in Problem 37, which I inserted in brackets in the preceding quotation from Gardiner's *Grammar*, a procedure similar to reducing fractional denominators to a common denominator is involved.[21] In fact, it is also used in the table preceding the one in the proof. Let us review the whole problem. It involves the finding of an unknown when (3 + 1/3 + 1/9 + 1/9) times the unknown equals 1. First he shows that the sum in parentheses multiplied by 1 sums at 3 1/2 1/18, which is equivalent to first assuming that the unknown quantity is 1. Then to find the value of the unknown quantity the following table is presented that calls up 1 by operating with 3 1/2 1/18, i.e., by dividing 1 by 3 1/2 1/18:

Call 1 out of 3 1/2 1/18.

1	3 1/2 1/18
1/2	1 1/2 1/4 1/36
\ 1/4	1/2 1/4 1/8 1/72
1/8	1/4 1/8 1/16 1/144
1/16	1/8 1/16 1/32 1/288
\ 1/32	1/16 1/32 1/64 1/576
Total:	1 [for if we] add

1/2 1/4 1/8 1/72 1/16 1/32 1/64 1/576 [we get 1. Now]

8 36 18 9 1

[are the values of the smaller fractions under which they are written when taken as parts of 576. These parts] total **72** [which is] 1/8 [of 576. Therefore the answer is 1/4 1/32].

That is to say, 1/4 1/32 is the value of the unknown quantity. This is shown by the author when he considers the smaller fractions beginning with 1/72 and ending with 1/576 as a series of parts of 576, the least common number, the one embracing all of the denominators of the smaller fractions. Those numbers of parts given here in bold type (i.e., rubricated in the papyrus, and hence may be called "red auxiliaries") add up to 72. But 72/576 is 1/8, and thus if we add the larger fractions 1/2 1/4 1/8 to the 1/8 just determined, we see that the list of fractions given in the total are just the fractions opposite 1/4 and 1/32. Thus the value of the unknown is 1/4 + 1/32. The reader will notice, if he turns to the Example of Proof in Problem 37, that the table showing the multiplication of the computed unknown (1/4 1/32) by the numbers of the "unknown quantity" specified at the beginning of the problem, namely 3, 1/3, 1/3 of 1/3, and 1/9, employs the same technique of converting the denominators of the smaller unit fractions into integer parts (or red auxiliaries) of the common denominator (288) in a manner precisely like that described in the table given before it, which we have reproduced above. This same sort of use of a common denominator and the red auxiliary numbers is seen in the completion Problems 21-23 of Document IV.1, and in Problem 36 where the author sums 16 [!] unit fractions as parts of a common denominator (1060) and in Problem 38 where he sums 7 unit fractions as whole number parts (or red auxiliaries) of a common denominator (66). We should note that the parts as applied to common denominators may be as integers, or numbers comprised of integers and unit fractions, sometimes written in red and sometimes not. Good examples appear in the completion Problems 7A, 7B, 8, 13-15, and 19-20 of Document IV.1, though the whole group of completion problems 7A-20 involves common denominators. I reproduce here only Problem 7A:

[Multiply 1/4 1/28 by 1 1/2 1/4.]

Example of completion (*tp n skmt*):

1	1/4 1/28 [as parts of 28 these are] **7 [and] 1**
1/2	1/8 1/56 [as parts of 28 these are] **3 1/2 [and] 1/2**
1/4	1/16 1/112 [as parts of 28 these are] **1 1/2 1/4 [and] 1/4**
Total:	1/2 [since as a part of 28 this is 14].

It is obvious to the reader that the bold (i.e., red) auxiliary numbers written in the right column represent the unit fractional sums as parts of 28, and that those numbers include both integers and unit fractions, as do the red numbers given in the division of 2 by 7 given below in note 35.

Having explored some of the fundamental Egyptian arithmetic procedures applying to multiplication, division, and the addition of unit fractions with or without the use of red auxiliaries, we should now examine the Table of Two, occupying the first part of the Rhind Papyrus (for the hieratic text of the table and its hieroglyphic transcription see Figs. IV.2a [Plate 2]-IV.2q [Plate 33]).

A fair question raised by modern students of the table was whether there was some general formula or formulas that were used to accomplish the various divisions of 2 by the odd numbers from 3 to 101. The consensus of investigators seems to be that there were no general formulas used by the Egyptians to construct the table and hence that there were instead sets of trial procedures used, as we shall see shortly. But the problem of constructing the table has always appealed to mathematicians, who have given interesting modern interpretations (even involving computer generations of possible alternative solutions of the table) but hardly ones that add much to the actual empirical construction of the table given in the Rhind Papyrus. Of the extensive literature on the Table of Two I shall discuss or quote from only a strictly limited number of works produced over the last century and one-quarter, i.e., since the first publication of a facsimile edition of the Rhind Papyrus in 1871 by Eisenlohr.[22]

A good place to begin our discussion of the Table of Two is with F. Ll. Griffith's treatment of it in 1894, a few observations from which I now quote, with my additions in brackets:[23]

> Since the Egyptians possessed no expressions for fractions with a numerator above unity [except, of course, 2/3 and 3/4], they were compelled to exercise their ingenuity in order to make the root [or unit] fractions (*Stammbrüche* in German)[24] serve the same end. They were not satisfied with such clumsy expressions as 1/15 + 1/15 + 1/15 + 1/15 + 1/15 + 1/15 + 1/15 for 7/15: considering the notion of 7/15 as the division of 7 by 15, they could have reckoned 5÷15 = 1/3, 2 remaining to be divided by 15: this latter would then be found to be equivalent to 1/10 + 1/30, so that the *notion* 7/15 could be *expressed* as 1/3 + 1/10 + 1/30.
>
> Now it has been pointed out by Professor Cantor that *any* simple fraction can be resolved into *Stammbrüche* by subdivision into *Stammbrüche* (1-fractions [i.e., fractions

with numerator 1]) and 2-fractions [i.e., fractions with numerator 2] (given that 2/11 = 1/6 + 1/66, 5/11 = 1/11 + 2/11 + 2/11 = 1/11 + 2/6 + 2/66 = 1/3 + 1/11 + 1/33), and that the Egyptians became aware of this.

It was to supply the want of a 2-series, and to resolve these 2-fractions into...1-fractions, that the Egyptians formed tables of the division of 2, expressing, *e.g.*, the division of 2 by 13 not as 2/13 but as 1/8 + 1/52 + 1/104: divisions by the odd numbers alone were required, for 2 divided by an even number could be reduced at once to the 1-series. From Kahun there is a table of the simplest kind, reaching to 2÷21 [see Document IV.3 below, *A Table of Two*], but the first table in the Rhind Papyrus is carried as far as 2÷101 [corrected out of 99; see Document IV.1, last entry in the Table of Two]....

A good series of solutions for the lower numbers might have been obtained from the formula (put into algebra)—

$$2/n = 1/a + 1/na, \text{ where } a = (n+1)/2$$

[the $2/n$ and $1/a$ terms are later corrections by Peet out of the misprints z/n and $1/(a/2)$],...

An easily used formula is of great value in calculation, but for this stereotyped table the Egyptians made a wise selection from the possible values without being bound by any formula.

The author of the table undoubtedly chose the values that could be most readily utilized according to his system of dealing with fractions, a system which was founded on the [basic] multiplication of whole numbers [i.e., by doubling, multiplying by 10, etc.], and on division by 1/2, starting from 1 or 2/3; only in cases of necessity using the cumbrous system of a greatest common measure [i.e., the so-called red auxiliaries, which in fact were not always the "greatest" common denominator, as has been noted above].

In pursuing the fundamental empirical procedures used in proving and presenting the various divisions of 2 by the odd numbers beyond the analyses and judgments given by Griffith, we should like to examine the crucial textual work of two other students of the Rhind Papyrus, both of whom, like Griffith, had superior knowledge of the language along with a thorough understanding of the literature on Egyptian mathematics produced in the first half-century after the original publication of the Rhind Papyrus by Eisenlohr. The first of them is T. Eric Peet.[25] Peet first presents the entry for 2+7 as given in the Table of Two, which I have rendered as follows in Document IV.1 below:

1/4 [of 7 is] 1 1/2 1/4, 1/28 [of 7] is 1/4.

1	7			
1/2	3 1/2		1	7
\ 1/4	1 1/2 1/4		2	14
\ 4	28	1/4	4	28.

Then Peet comments [with my additions]:

> There is no doubt as to what takes place here. The 2 is broken up into two parts, namely (1 1/2 + 1/4) and 1/4. The first of these is then shown to be 1/4 of 7 by the simple process of dividing 7 by 2 and then by 2 again, while the second is shown to be 1/28 of 7 by multiplying 7 by 4 and obtaining 28 [and then using its reciprocal, though Peet fails to state the obvious].
>
> As a proof this is satisfactory, but it does not throw the slightest light on the one feature of interest in these problems, namely the manner in which the Egyptian obtained his answer, which, be it noted, is not worked out at all, but merely assumed and then proved. To arrive at the method by which the answer was obtained it is necessary to examine the whole series of resolutions from 2/3 to 2/101, and to try to discern in them any signs of the employment of a general formula....

In the case of all... fractions whose denominator is a multiple of 3 [other than 2/3 itself which is not reduced] a very obvious resolution presented itself, for the numerator 2 could be broken up into 1 1/2 and 1/2, and since 1 1/2 divides exactly into 3 and all its multiples the problem was at once solved. Thus: 2/9 = (1 1/2 + 1/2) / 9 = 1/6 + 1/18 [the 1/18 being determined by taking the reciprocal of the double of 9].

Similarly those fractions whose denominator was 5 or a multiple thereof could be dealt with by breaking up the numerator 2 into 1 2/3 and 1/3. Thus 2/25 = (1 2/3 + 1/3) / 25 = 1/15 + 1/75. In this way the denominators 5, 25, 65, and 85 were dealt with, 15, 45 and 75 having been already treated as multiples of 3, 35 being treated irregularly, 55 as a multiple of 11, and 95 as a multiple of 19.

When the denominator was divisible by 7 the 2 was broken up into....1 + 1/2 + 1/4...and 1/4....In this way were resolved 7, 49 and 77; 21 and 63 were treated as multiples of 3, and 35 and 91 were dealt with irregularly.

In the case of 11 and its multiple 55 the 2 was resolved into 1 2/3 + 1/6....and 1/6.

Up to this point it may be said that the method has been marked by considerable regularity. We are now left with the prime numbers between 13 and 97. A modern mathematician would probably treat these, as indeed all numbers, by some such formula as that suggested by Griffith [given above in my quotation from his article]....which has the advantage of resolving each 2-fraction into two aliquot parts only, but the disadvantage of giving a second fraction with a very high denominator. The Egyptian was bound by no fetters of this kind, for the simple reason that he reached his results not by formula but by trial. An inspection of them is sufficient to show this. Even in the treatment of multiples of the lower prime numbers we have already seen that there was some irregularity, and this is only emphasized when we come to the higher prime numbers.

Eisenlohr has attempted to embrace the Egyptian results [concerning the Table of Two] under a series of rules which he enunciates as follows:

1. Resolution into three fractions was preferred to resolution into four.

2. If a resolution existed (i.e. could be found) in which the denominator of the first root-fraction was the product of factors which when separately multiplied by the denominator of the original 2-fraction give the denominators of the remaining root-fractions, if, that is to say, $2/n$ could be broken up into $1/ab + 1/an + 1/bn$ or into $1/abc + 1/an + 1/bn + 1/cn$, then this resolution was chosen. Otherwise that in which the denominator of the first root-fraction was not ab but $ab/2$, $ab/3$, $ab/4$, or $ab/8$ was adopted.

3. High factors of the original denominator were avoided.

It is true that as a matter of actual fact the resolutions given by the Egyptian do to some extent conform to these rules. Thus the resolutions of 17, 31, 37, 43, 47, 59, 67, 73 and 97 conform to the simple formula given above; in the case of 19, 41, 71, 79, and 83 the denominator of the first root-fraction is $ab/2$, in the case of 53 it is $ab/3$, in the case of 13, 29, and 89 it is $ab/4$, and in the case of 61 it is $ab/8$. But even here Eisenlohr's rules are by no means consistently carried out....

The fact is that Eisenlohr is here employing a method of analysis which ought not to be applied to Egyptian mathematics. Even could we show that all the results corresponded to a formula this would not prove that the Egyptian worked by this formula, and if Eisenlohr means to imply that he did he is undoubtedly wrong. The Egyptian, far from employing a formula, probably had no conception that the resolution could be accomplished by a single method in all cases. He had indeed observed that where the denominator of the 2-fraction was a multiple of 3 the same resolution could be used as for 3 itself, and similarly for 5,

> 7, and even 11; but when it came to the higher prime numbers he had no formula to help him.
>
> His method was undoubtedly that of trial. He had grasped the fact that the problem consisted in breaking up 2 into the sum of several quantities each of which would divide without remainder into the given denominator....

Peet goes on to detail the trial methods and gives finally a table that "will enable both the results and the method employed in obtaining them to be seen at a glance." This table I have given as Fig. 28. But, in fact, one needs more than a glance. And thus other students of the Table of Two have gone to great length to detect the various trial procedures used by the Egyptian author.

One of these authors is A.B. Chace whose edition is superb and whose explanation of the procedures employed by the author, while not quite so analytical as some of the later treatments of the hypotheses proposed for the construction of the Table of Two described below, is still well worth examining:[26]

> In reproducing the table [as given on pp. 21-22 = my Fig. IV.29] I have marked the different cases A, B, AD, BD, C and E; that is, I have used:
>
> A when the author first takes 2/3;
>
> B when he simply halves;
>
> D along with A or B when he also uses 1/10 or 1/7;
>
> C when at some step he gives a whole number and uses its reciprocal as a multiplier;
>
> E for the three special cases of 35, 91, and 101.

Following Chace I shall illustrate each type of procedure by examining one example, except for the longer descriptions of the procedures in the cases of the divisions of two by 35, 91, and 101, which are somewhat longer and I leave to the reader. For case A (involving the taking of 2/3 as the first step) let us look at 2/17, the first problem in the table for which the method is completely given:

[2 divided by 17] [cont.]

Call 2 out of 17 [i.e., Get 2 by operating on 17].
1/12 [of 17 is] 1 1/3 1/12, **1/51** [of 17 is] 1/3, **1/68** [of 17 is] 1/4.
Procedure:

1	17				
2/3	11 1/3				
1/3	5 2/3		\ 1	17	
1/6	2 1/2 1/3		\ 2	34	
\ 1/12	1 1/4 1/6	[Total:]	3	51	1/3
Remainder[27] 1/3 1/4			4	68	1/4.

Its evident that we must find multipliers of 17 that will produce products that add up to 2. We first try 2/3 as a multiplier and we have of course far exceeded the required 2, which we are trying to call up out of 17. Hence, the author indicates, we must take successive halves, namely 1/3, 1/6, 1/12. With 1/12 he arrives at a product that is less than 2, namely 1 1/4 1/6. This is short of 2 by the amount 1/3 1/4, which he calls the remainder. To find 1/3 of 17, the author multiplies 17 first by 1 and then by 2, which products added together equal 51. Then, taking the reciprocal of 51, i.e., 1/51, and multiplying it by 17 the author gets 1/3. In precisely the same way, the author finds that 1/68 of 17 is 1/4. Accordingly the full division has been made, and the complete answer is 1/12 1/51 1/68. The method of finding the required remainder, namely 1/3 1/4, is thus like that given in the completion problems (21-23) of Document IV.1.

As an illustration of the second kind of procedure, that is, B where the first step is to use a multiplier of 1/2, is the division of 2 by 13. First I give my version of the division:

[2 divided by 13]
1/8 [of 13 is] 1 1/2 1/8, **1/52** [of 13 is] 1/4, **1/104** [of 13 is] 1/8.

1	1[3]	
1/2	6 1/2	
1/4	3 1/4	
\ 1/8	1 1/2 1/8	
\ 4	52	1/4
\ 8	104	1/8.

As in type A, we see here that after successive halvings of the first fractional multiplier (1/2 here), we reach 1/8, which yields, as a product less than 2, 1 1/2 1/8. The remainder in this case to make 2 is 1/4 and 1/8. As before the multipliers to produce the remainder are found by taking the reciprocals of the products given by multiplying 13 by 4 and 8 respectively, that is 1/52 and 1/104.

Procedure AD is exemplified by the division of 2 by 25, which I give from Document IV.1 below:

[2 divided by 25]
1/15 [of 25 is] 1 2/3, **1/75** [of 25 is] 1/3.

1	25	
\ 1/15	1 2/3	
\ 3	75	1/ 3.

Here the author probably first took 1/10 of 25, i.e., 2 1/2, and then 2/3 of that, resulting in 1/15 of 25, which is thus 1 2/3. It is obvious that this was short of 2 by 1/3, which he determined by taking the reciprocal of 3 x 25, or 1/75. This gave him as the final answer or quotient stated in the first line, namely 1/15 1/75.

Procedure BD is represented by the division of 2 by 31, presented from Document IV.1 below:

[2 divided by] 31
1/20 [of 31 is] 1 1/2 1/20, **1/124** [of 31 is] 1/4, **1/155** [of 31 is] 1/5.

1	[31]	
\ 1/20	1 1/2 1/20	
\ 4	124	1/4
\ 5	155	1/5.

The first, unspecified fractional multiplier was obviously 1/10, which would give a product of 3 1/10. Then taking half of that multiplier, i.e., 1/20, the product as given is 1 1/2 1/20, the first product below 2. The remaining multipliers to add up to 2 are 1/4 and 1/5. Then, as in the other examples, we find the remaining unit fractions by taking the reciprocals of 4 x 31 and 5 x 31, thus obtaining the unit fractions 1/124 and 1/155, and hence producing the complete answer 1/20 1/124 1/155.

As an example of procedure C, Chace analyzes 2 divided by 21, as given in Document IV.1 below:

[2 divided by 21]
1/14 [of 21 is] 1 1/2, 1/42 [of 21 is] 1/2.

1	21	
\ 2/3	14	1 1/2
\ 2	42	1/2.

This can hardly be conceived of as a separate category, except that all of the products are whole numbers, and as usual we find the final unit fractions by taking their reciprocals. As Chace points out, 21 is a multiple of 3, and thus 2/3 of 21 is 14 and hence 1/14 of 21 is 1 1/2. The remainder to add up to 2 is 1/2. Then the reciprocal of 2 x 21 is 1/42. Hence the complete quotient is 1/14 1/42. Chace analyzes the division of 2 by 65 in a similar way, noting that 65 is a multiple of 5. As I have said earlier, I leave Chace's analysis of the special cases of the divisions of two by 35, 91, and 101 to the reader's perusal. But I find his final comments on the Table of Two to be of interest:[28]

> All of these various cases seem to indicate that there was no definite rule for determining the multipliers to be used, but probably the slow experience of different writers suggested different multipliers for different examples, as they seemed to them the easiest or gave results in the most satisfactory form.
>
> In the table as here reproduced [see Fig. IV.29] I have put: first, the letter or letters indicating the kind of multipliers employed; second, the number [i.e., the odd-number divisor of 2]; third, the first fraction of the answer, this being the multiplier that produces a number a little less than 2; then the number a little less than 2 that is produced

by this multiplier; the remainder necessary to make 2; and finally, the answer.

To complete this discussion of the Table of Two, I wish to illustrate the opinions of more recent students, centering on the treatment of the Table of Two by B.L. van der Waerden. It will be noticed that the author refers to the hypotheses of Neugebauer and Vetter.[29] Now for some of the highlights of van der Waerden's paper of 1980.[30] After detailing the fundamental Egyptian arithmetical procedures we have already described in some detail, he succinctly comments on (pp. 262-63) the use of auxiliary numbers (I have called them here the "red auxiliaries"):

> In order to perform a division it is sometimes necessary to calculate the supplement that must be added to a given sum of unit fractions in order to obtain 1. In the example (2:17) just explained [in the account explaining Chace's category A] the given sum of unit fractions was 1/4 + 1/6, and the supplement 1/3 + 1/4. In this case it was easy to see that 1/4 + 1/6 plus 1/3 + 1/4 is 2, but in higher cases it is not so easy to find the required supplement. Therefore, a whole section of the Rhind Papyrus is devoted to this problem. For its solution, *auxiliary numbers* were introduced. They were also used to check whether a given sum of fractions is equal to another sum of fractions.

Before going on to analyze the Table of Two, van der Waerden points out that the Egyptian mathematicians used another method of establishing equalities. This is found in the Mathematical Leather Roll of the British Museum, which we give as Document IV.5. From the equalities given there van der Waerden lists the following two sequences:[31]

1/9 + 1/18 = 1/6	1/14 + 1/21 + 1/42 = 1/7
1/12 + 1/24 = 1/8	1/18 + 1/27 + 1/54 = 1/9
1/15 + 1/30 = 1/10	1/22 + 1/33 + 1/66 = 1/11
1/18 + 1/36 = 1/12	[1/26 + 1/39 + 1/78]* = 1/13
	*corr. by v.d.W. from *[cont.]*

1/28 + 1/49 + 1/ 196
1/21 + 1/42 = 1/14 1/30 + 1/45 +1/90 = 1/15.
1/24 + 1/48 = 1/16

Van der Waerden notes that the sequence on the left derives from the equality of 1/3 + 1/6 = 1/2, labeled (1), (which is often used in the Rhind Papyrus) by dividing it by 3, by 4, by 5, by 6, by 7, and by 8. The second sequence can be derived in the same way from the equality

$$1/2 + 1/3 + 1/6 = 1. \qquad (2)$$

"We shall see that the same method of deriving sequences of equalities from one simple equality was used in the (2:*n*) table of the Papyrus Rhind to obtain the results of divisions 2:*n* in all cases in which *n* is a multiple of 3." After his preliminary remarks on the procedures with fractions, van der Waerden, like most authors treating the Table of Two, categorizes the various entries into 5 groups according to the pattern used:[32]

> A. The 2/3-group. It consists of those divisions in which *n* is a multiple of 3. They all follow the same pattern:
>
> $$2:3m = 1/2m + 1/6m.$$
>
> B. The *division group*. The calculations in this group are just standard divisions...in which either the 2/3 - sequence [i.e., 2/3 1/3 1/6 ...] or the 1/2 - sequence [i.e., 1/2 1/4 1/8 ...] is used. The results of these divisions all have the same form
>
> $$2: n = 1/x + 1/k_1n + 1/k_2n + 1/k_3n \qquad (3)$$
>
> with 2 or 3 or 4 terms on the right, *x* being an integer between 1/2*n* and *n*. If the 2/3 - sequence is used, *x* is 3 or 6 or 12 or 24, but if the 1/2 - sequence is used, *x* is a power of 2.
>
> Examples:
>
> 2:5 = 1/3 + 1/15
> 2:7 = 1/4 + 1/28
> 2:11 = 1/6 + 1/66
> 2:13 = 1/8 + 1/52 + 1/104.

To this group [then] belong the divisions (2:n) with $n = 5$, 7, 11, 13, 17, 19, 23, 29, 37 and 41.

C. The *derived group*. The divisions of this group can be derived from those of group B by multiplying all denominators by 5 or 7 or 11 or 13 or 17. Thus, from

2:5 = 1/3 + 1/15

one derives

2:25 = 1/15 + 1/75.

This group contains the cases n = 25, 49, 55, 65, 77, 85, 95.

D. The *algorithmic group*. In this group the results of the divisions also follow the pattern [of] (3), but the denominator x is neither 2^k nor 3 times 2^k. The divisions of this group can only be understood by assuming the use of the algorithm of auxiliary numbers.

E. Three exceptional cases remain, namely $n = 35$ and $n = 91$ and $n = 101$. In the last case a "trivial" decomposition is given:

2:101 = 1/101 + 1/202 + 1/303 + 1/606.

In the former two cases the results of the divisions cannot be written in the form (3).

But van der Waerden gives the results of these divisions assigned to group E, needless to say, as presented below in Document IV.1. He notes that of the three it is only in the case of 2:35 that auxiliary numbers are given in the papyrus, and he goes through the steps given in the text. Having thus presented the patterns of the 5 groups of divisions found in the 2:n table, van der Waerden then succinctly and clearly presents the hypotheses proposed by Neugebauer, Vetter, and Vogel (in various of their accounts cited in note 29) and their motives for proposing such hypotheses. This account I leave to the reader, quoting here only van der Waerden's concluding remarks.[33]

The kernel, from which the calculating apparatus of the Rhind Papyrus was developed, was the algorithm of Egyptian multiplication, in which the main operation was doubling. Now if a unit fraction, say 1/7, is doubled, one gets 1/7 + 1/7, and if this is doubled once more, an unwieldy expression like 1/7 + 1/7 + 1/7 + 1/7 is obtained. To avoid this, the Egyptians devised a method to rewrite [it] as a sum of different unit fractions, e.g.

$$1/7 + 1/7 = 1/4 + 1/28$$

which can be doubled again and again without any difficulty.

The Egyptian solution of this rewriting problem, codified in the (2:*n*)-table, is based on three fundamental ideas:

First idea: If you want to rewrite $1/n + 1/n$ as a sum of fractions, you should ask: What part of *n* is 2? A standard division of 2 by *n* will give you the answer.

Second idea: From the obvious equality

$$1/3 + 1/3 = 1/2 + 1/6$$

you may obtain a whole sequence of equalities

$$1/3m + 1/3m = 1/2m + 1/6m.$$

Just so, from

$$1/5 + 1/5 = 1/3 + 1/15 \text{ and}$$

$$1/7 + 1/7 = 1/4 + 1/28$$

you may obtain results like

$$1/25 + 1/25 = 1/15 + 1/75.$$

Third idea: If you want to complete a sum of fractions to 1 or to rewrite a sum of fractions as another sum, you may multiply all given fractions by a conveniently chosen integer *D*, and afterwards divide again by D. This is the method of "auxiliary numbers", which was certainly applied in the case [of] 1/35 + 1/35, and probably in many other cases.

The first idea was strongly stressed in the whole arrangement of the (2:*n*)-table. The question "What part of *n* is 2?" was placed as a heading above every section of the table [or in fact its equivalent "Call 2 out of *n*"], even in cases in which the division 2:*n* was only a verification afterwards [i.e., a proof of the answer given at the beginning of each entry].

> The second idea, the idea of serial derivation of a whole sequence of equalities, is made clear by many examples in the Leather Roll [i.e., our Document IV.5].
>
> The third idea is applied to many examples in the sequel of the Rhind Papyrus [i.e., the part of Document IV.1 after the 2:*n*-table]. Thus, it appears that the Papyrus as a whole is a unity.
>
> Following Peet, I formerly thought that the (2:*n*) table was developed in the course of a long historical process, possibly during many centuries. But now...I am more inclined to believe that the whole papyrus is the work of one man, who was at the same time an ingenious arithmetician and a very good teacher.

Bruins in the article mentioned in note 29 (p. 81) had previously argued the slightly different conclusion that "an Egyptian mathematician, in accordance with the arithmetic of his days, was able to construct the table within a very short time, in any case within a day!"

In concluding this account of the construction of the Table of Two, I shall add one more opinion concerning the criteria that the Egyptian mathematician followed in selecting the unit fractions that constituted the quotients making up the table. This comes from the perceptive account of Egyptian mathematics given by R.J. Gillings:[34]

> Opinions are quite varied on the precepts, standards, or tests by which the scribe was guided in his choice of values from the hundreds available to him. Some previous investigators have attempted to give the scribe's possible precepts. I here present the five precepts which I believe were the scribe's primary guide. The fifth precept has not been suggested, to my knowledge, by any previous writer...

PRECEPT 1 *[cont.]*

> Of the possible equalities, those with the smaller numbers are preferred, but *none* as large as 1000 [890 being the largest number in the table].
>
> PRECEPT 2
>
> An equality of only 2 terms is preferred to one of 3 terms,[35] and one of 3 terms to one of 4 terms, but an equality of more than 4 terms is *never* used.
>
> PRECEPT 3
>
> The unit fractions are always set down in descending order of magnitude, that is, the smaller numbers [i.e., denominators] come first, but *never* the same fraction twice.
>
> PRECEPT 4
>
> The smallness of the first number [denominator] is the main consideration, but the scribe will accept a *slightly* larger first number, if it will *greatly* reduce the last number.
>
> PRECEPT 5
>
> *Even* numbers are preferred to *odd* numbers, even though they might be larger, and even though the numbers of terms might thereby be increased.

Other rules and tables, some existing and some hypothesized by Gillings and others, are described in the next section and should throw further light on the calculating techniques exhibited in the Table of Two. A prime example would be the table or tables of multiplications by two-thirds mentioned as possibilities in the next section.

Other Tables in Aid of Calculation

Following the Table of Two in Document IV.1, the author inserts a table of the division of the first 9 units by 10. Though we are not told how it was developed, it is easy to see how the ordinary techniques of division would have easily produced it. Division of 1 by 10 is immediately obvious if we multiply 1 by 10 and then take the reciprocal, namely 1/10. The division of 2/10 is equally evident, since 1/10 and 1/10 yields 1/5, as the author could have seen from a common table of equalities like that in Document IV.5

(see Col. 2, line 4). The division of 3/10 is obviously the sum of the divisions of 1/10 and 2/10. The division of 4/10 is equivalent to the division of 2/5, whose solution is given in the Table of Two. The division of 5 is immediately evident as 1/2. The rest of the entries can be easily worked out by the sums of the various entries in the first half of the table.[36] This table allows the quick solution of the first six problems of the document, namely the division of 1, 2, 6, 7, 8, and 9 loaves among 10 men. Notice that the divisions of 3, 4, and 5 loaves among the 10 men are missing. Perhaps the author thought their solutions to be so obvious that they need not be included. It should be remarked that in Problems 1-6 the given answer is proved by the formal multiplication of the given answer by multipliers that add to 10, thereby producing products that add up to the given number of loaves being divided.

We should also notice that some such table as this one of units divided by 10 would be useful in the building procedure of determining the man-days of laborers needed for removing materials, since it always involves the division of the volume of material by 10, the amount that a single man was assumed to remove in a day. I have suggested how this table might have been used to find approximations to the sums of fractions in Document IV.6 (i.e., Reisner Papyrus 1, Sect. G, lines 10-11 and 14-15), where, in the first example, the division of 39 by 10 is approximated at 4, perhaps by rounding off the exact Egyptian way of expressing 3.9, namely, 3 2/3 1/5 1/30, the fractional part of which could have been quickly obtained from the table.

Inserted as Problem 61 of Document IV.1 is a table of the multiplication of fractions, that, at the least, is not in a coherent position in the document. The first nine lines give the following multiplications: 2/3 of 2/3 is 1/3 1/9; 1/3 of 2/3 is 1/6 1/18; 2/3 of 1/3 is 1/6 1/18; 2/3 of 1/6 is 1/12 1/36; 2/3 of 1/2 is 1/3; 1/3 of 1/2 is 1/6; 1/6 of 1/2 is 1/12; 1/12 of 1/2 is 1/24; 1/9 of 2/3 is 1/18 1/54 or [putting it in another way] 1/9, 2/3 of it [i.e., 1/9] is 1/18 [1/54]. After a break in the text, the last five lines of multiplications are in the different form found at the end of the ninth line: 1/5, 1/4 of it is 1/20; 1/7, 2/3 of it is 1/14 1/42; 1/7, 1/2 of it is 1/14; 1/11, 2/3 of it is 1/22 1/66 and 1/3 of it is 1/33; [and] 1/11, 1/2 of it is 1/22 and

1/4 of it is 1/44. The use of a second form of presentation seems to remind the reader that the only "legitimate multipliers are 2/3 and 1/2, and fractions obtained from them by halving."[37] In the problem arbitrarily numbered 61B, which follows this table, is a general rule for taking 2/3 of an odd fraction, i.e., of the reciprocal of an odd number: take the reciprocals of the products of 2 times the odd number and of 6 times the odd number, and these two unit fractions together give the desired solution. But as the reader will see, the author only illustrates the rule by taking the odd-number 5. Since that solution is equivalent to the quotient of the division of 4 by 30, i.e., 2 by 15, it is evident by the Table of Two that that division is equal to 1/10 1/30, when expressed in the Egyptian manner.

The great frequency of the use of 2/3 as a multiplier of both fractions and whole numbers displayed in the documents (and particularly in Document IV.1[38]) has prompted scholars to pose that the Egyptians prepared multiplication tables for multiplications by 2/3. Indeed the table of Problem 61 analyzed in the preceding paragraph seems to indicate that such was the case for multiplying the commonly used unit fractions by 2/3. But let us look at the views of two proponents of the view that the Egyptians prepared tables of 2/3 of both fractions and whole numbers, first to Peet's argument:[39]

> Strange as it may seem to us, the Egyptian was accustomed to take two-thirds of a number by a single process. No doubt he used tables for the purpose, but the mere fact that tables existed is one more testimony to the fundamental nature of the concept of "the two parts" in the Egyptian mind. Most of us when asked to give two-thirds of 5 would do it by taking one third and doubling it. So far was the Egyptian from doing this that his sole means of finding one-third of a quantity was to take two-thirds and then halve it.

But Battiscombe Gunn in his probing review of Peet's edition of the Rhind Papyrus doubts Peet's conclusions regarding the use of a 2/3 table:[40]

> That the Egyptian reckoner used tables for the purpose of taking two-thirds of a number seems to me quite doubtful. In the first place, no such tables are known before Byzantine times; we might well have expected one in *Rhind*, if it were necessary. Secondly, it was perfectly easy for a reckoner to take two-thirds of any number below 100 in his head, by splitting up the number in question into not more than three parts, provided that he knew by heart (as no reckoner could fail to do) the 2/3 of about a dozen numbers, say of 2, 3, 4, 5, 6, 15, 21, 30, 45, 60, 75, 90, of which all but 4 and 5 are chosen for their obviousness. Thus, 2/3 of 87 is 2/3 of 75, 9 and 3; of 50, 2/3 of 45 and 5; of 28, 2/3 of 21, 6 and 1; and so on. Anyone will find that with a few minutes' practice he can do this mentally with ease. In some cases there are short cuts by subtraction; *e.g.*, 2/3 of 29 is not only 2/3 of 21, 6 and 2 but also 2/3 of 30 less 2/3 (of 1)....Taking two-thirds of numbers higher than 100 by the same process is only a matter of extending one's repertory.
> [Peet's statement:] "His sole means of finding one-third of a quantity was to take two-thirds [of it] and then halve it"....is surely an overstatement. The fact that halving two-thirds is his almost invariable method of *arriving at* one-third on paper must not blind us to the other fact that the Egyptian, like everyone else, had ultimately no way of arriving at two-thirds but *via* one-third. I think that he used this roundabout method for two reasons: first, that it was less trouble to acquire facility in taking two-thirds of a number, and to halve this when one-third was wanted, than to acquire the practice of taking both two-thirds *and* one-third mentally; secondly, that the method has this great advantage, that you can easily check your mental arithmetic by adding the two-thirds and the one-third together and seeing if the total equals the number operated upon....

This criticism of Peet's almost casual acceptance of the Egyptian use of two-thirds tables appears to have some merit. But if we consider the 2/3 and 1/3 entries in the table of Problem 61, discussed above and elsewhere in the Table of Two and in other problems of the Rhind Papyrus that might have been drawn from two-thirds and one-third tables, and the great convenience of having such tables, we can understand Peet's conclusion. Still it has to be underlined, as Gunn notes, that no such complete tables for either fractions or whole numbers are known before Byzantine times. The reader may consult in our Fig. IV.30 the 2/3 table that appears in the Greek in the first part of the *Papyrus mathématique d'Akhmîm* about 1000 years after the Rhind Papyrus that comprises our Document IV.1. Obviously that table can hardly be used as evidence for the existence of ancient Egyptian tables, despite the very extensive Egyptian influence on the much later Greek work. The inclusion in the Byzantine work not only of a table of two-thirds but of the succeeding tables of multiplications by fractions from 1/3 through 1/20 is nowhere matched in the early Egyptian documents. Nevertheless, R.J. Gillings discusses how 2/3 tables of fractions (as well as similar 1/3 and 1/2 tables of fractions) might have been prepared, if we accept the possibility that such tables were in existence at the time of the Rhind Papyrus.[41] In reconstructing these supposed Egyptian tables, Gillings laces together individual line solutions largely found in the Rhind Papyrus. See Fig. IV.31 below for Gillings' reconstructed fractional tables in the Egyptian manner.

Three more tables are found in Document IV.1. All three are concerned with volumetric measures. The first one is numbered Problem 47 in Document IV.1 and comprises the Division of 100 Quadruple Heqat by 10 and its succeeding 9 multiples. The results are given in quadruple heqat and its Horus-eye fractions *(1/2, 1/4, 1/8, 1/16, 1/32, 1/64)* and additional *ro* and fractions of *ro*. It will be remembered from our earlier discussion of volumetric measures that 1 *ro* = 1/320 *heqat*. Appropriately this table is found in the section including problems concerning the contents of granaries and their dimensions. The final tables from Document IV.1 are labeled Problems 80 and 81. The first of the two simply gives successively the values in *henu* of 1 *heqat* and then its Horus-eye fractions. The second is more complex. It repeats first the equivalence in *henu* of the simple Horus-eye fractions as given in the preceding table and then follows that with a series of sub-tables that give the values in *henu* of decreasing sums of Horus-eye fractions plus some *ro* fractions, with all but the first sub-table giving, in an extra column to the right, the values of the Horus-eye sums in rubricated regular fractions of a heqat.

We should also remind the reader once more of the table of equalities expressing regular fractional sums that constitute Document IV.5. The probable role they played in the composition of the Table of Two has already been discussed in the preceding section. Note that on the basis of the equalities given in lines 11, 13, 19 of column 3 and lines 1-7 of column 4, Gillings poses a hypothetical G rule, which was not in any Egyptian document so far as I know. But still Gillings' preliminary remarks concerning the G rule are of some interest and I give them in brief:[42]

> An intelligent scribe would certainly notice a certain simple relation existing between the three terms of these equalities [mentioned above]. The expression of this relation is the G rule. In modern mathematical terms we may state it as follows:
>
> *G rule:* If one unit fraction is double another then their sum is a different unit fraction *if and only if* the larger denominator is divisible by 3. The quotient of the division is the unit fraction of the sum.
>
> But if such a rule were expressed by an Egyptian scribe, it would have been much terser, probably something like this:
>
> For adding 2 fractions, if one number is twice the other, divide it by 3.
>
> Line 11....[of column 3 of Document IV.5] illustrates the G rule: 1/9 1/18 = 1/6.[43]

Gillings goes on to discuss possible extensions of the rule, without of course claiming that either the rule or its extensions are found in the ancient Egyptian mathematical literature; I leave Gillings' treatment of those extensions to the reader's perusal.[44]

Gillings also displays an extensive table of Two-Term Equalities (in the form of 10 sub-tables), which is not present in

Egyptian documents and which we again leave to the reader to explore.[45] Furthermore, despite his recognition of the fact that no tables of squares or square roots have been found in the ancient Egyptian documents, Gillings shows how such tables might have been constructed and what they would look like.[46] Peet had earlier said that "though distinguished by a special name, whose literal meaning is not certain (apparently connected with the verb *sny*, to 'pass by') [squaring] was merely a special case of multiplication."[47] I quote the pertinent parts of Problem 11 of Document IV.2, where the squaring is indicated:

[Problem 11; see Fig. IV.6h]
[Col. XXI]
[Lin. 1] **Example of reckoning the work of a man in logs.**
[Lin. 2] If someone says to you: "The work of a man in logs;
[Lin. 3] the amount of his work is 100 logs
[Lin. 4] of 5 handbreadths section; but he has brought them in logs
[Lin. 5] of 4 handbreadths section." You are to square these 5 handbreadths. The result is
[Lin. 6] 25. You are to square the 4 handbreadths. The result is 16.
[Col. XXII]
[Lin. 1] Reckon with this 16 to get 25.
[Lin. 2] The result is 1+1/2+1/16 times. You are to take this number 100 times.
[Lin. 3] The result is 156 1/4. Then you shall say to him, "Behold,
[Lin. 4] this is the number of logs which he brought of 4 handbreadths section.
[Lin. 5] You will find that it is correct."

As the reader of the Column XXII of this problem will readily see, the scribe also was familiar with the use of a proportion to solve his problems. As Peet notes,[48] "A modern boy handling this sum would be expected to state the proportion 16 : 25 = 100 : x, and then either to multiply 100 by 25 and divide the result by 16, or, like the Egyptian, to divide 25 by 16 and multiply the result by 100" [though the latter would not use the symbolic letter x of a later algebra to express the unknown quantity, but merely call the unknown "a quantity" or "a number"]. We shall have more to say about proportions and the finding of unknown quantities in the next section.

Returning to squares and square roots, and having noted this example of finding squares and the special term for that multiplicative procedure, we should conclude this section with examples of taking the square roots of numbers. Note those appearing in the shorter fragment of the Berlin Papyrus 6619, i.e., Document IV.4, with the conventional word "corner" (*ḳnbt*) [i.e., "right angle" (⌈ in hieroglyphics) used for "square root"]:

> [Lin. 1] ...You should extract the square root of 6 1/4...[i.e., 2 1/2]
> [Lin. 2] ...[Take] this 2 1/2, which remains....[You take]
> [Lin. 3] ...[the square root of 400, i.e., 20]. Reckon [with 2 1/2 to obtain 20]...[The result is 8] times. [Multiply 8]
> [Lin. 4] ...[by 2 and 1 1/2.] You should [now] say to him, the square root[s]
> [Lin. 5] ...[of the component square]s according to this calculation (*irt*) [are 16]
> [Lin. 6] ...[and] 12. You say it is found ...[i.e., correctly?]

In the longer fragment of that same document (line 6) we find the square root of 1 1/2 + 1/16 (i.e., 25/16, when expressed as a modern improper fraction in order to show the immediately evident squares involved in the expression whose square root it is). The square root is correctly given as 1 1/4 (i.e., 5/4).

Aha Problems: The Finding of Unknown Quantities

Among the most intriguing arithmetical problems found in the ancient Egyptian mathematical documents are those that seek to find an unknown quantity when a sum of it and one or more of its parts is specified (this includes cases when the sum includes a negative term, or put more simply so as not to give it a too modern sound, when one or more parts is subtracted from the unknown quantity or from the unknown and some part of it). The obvious reason for heightened interest in these problems is that the Egyptian solutions of them anticipate (without of course actually using) techniques that later came to the fore with the invention of algebra. Let us first concentrate on the simple examples of such problems, namely those given in Problems 24-27 of the Rhind Papyrus (Document IV.1). They all involve the sum of an unknown quantity (lit. "heap," i.e., , ʿḥʿ) and some unit fraction of it (in fact, 1/7, 1/2, 1/4, or 1/5). In each case the unknown quantity is first assumed to be equal to the denominator of the unit fraction, and a false sum is calculated. Then by the use of the following proportion the true quantity is determined, that is, by calculating thus: "as many times as the calculated false sum must be multiplied to produce the true sum, so many times must the falsely assumed quantity be multiplied in order to find the true unknown quantity." Let us present Problem 26 as an example:

A quantity with 1/4 of it added to it becomes 15.

[Assume 4.] [That is] multiply 4, making 1/4, namely 1, [so that the] Total is 5 [proceeding in the usual manner:

\ 1	4
\ 1/4	1
Total:	5].

[As many times as 5 must be multiplied to make 15, so many times 4 must be multiplied to give the required number.]

Operate on 5 to find 15

\ 1	5
\ 2	10

Total: 3.

Multiply 3 times 4.

1	3
2	6
\ 4	12

This becomes 12. [And find its 1/4:]

1	12
\ 1/4	3
Total:	15.

[Hence] **the quantity is 12** and its 1/4 is 3 and **the total is 15.** [This checks out since the sum agrees with what was originally specified.]

As I have indicated in my version of Document IV.1, the solutions of these simple problems, while recording only arithmetical calculations, seem to use a technique like that of false position, that is first assuming a false value of the unknown, and then producing a corrected value by the above-noted use of a proportion. Now obviously if we refuse to accept the designation of these arithmetical methods as algebraic techniques because we are not using letters for known and unknown quantities and operating signs while expressing the problems as equations, and also if we are not generally using the rules and equations found in the modern manipulation of polynomial expressions and the factoring out of such expressions as coefficients of the unknown, then obviously we shall not find algebra as such in the Egyptian documents. But some early historians of mathematics would find in the Egyptian expression of equations and in their solutions of problems involving unknowns many of the nascent procedures that carry over into later algebra. Thus the great German historian of mathematics Moritz Cantor would find considerable identity between the Egyptian techniques and those of the later algebra. And Problems 24, 28-29, and 31 are just the examples he identifies as "nothing else than what the algebra of today calls equations of the first degree with one unknown." And he writes them all in the form of equations for comparison (these are essentially the same as the equations I have written for them in notes 17, 22, 29, and 31 of the Document IV.1, as the reader may readily confirm).[49] With this done, he then noted:

> The essence of an equation is far less in its wording *(Wortlaut)* than in its solution, and so in order to test the justification for our comparison, we must look at how Ah-

> mes [the scribe of the Rhind Papyrus] carries out his Hau- [i.e., Aha-]calculations. In doing so, he goes methodically to work, adding together the terms which one would say today stand to the left of the equal sign. To be sure, he does it in two ways, in one so that the unit fractions to be added stand next to each other as one single form, e.g., [Problem] No. 31: (1 2/3 1/2 1/7) x =33; in the other so that the actual addition is carried out by reducing to a common denominator, e.g., [Problem] No. 24: (8/7) x =10, No. 28: (10/9) x = 10; No. 29: (20/27) x = 10. In the first method mentioned, the coefficient of the unknown [expressed as 1 and some unit fractions] is divided into the given number....[in the Egyptian manner]; viz., in Nr. 31 one multiplies 1 2/3 1/2 1/7 until 33 results and finds the clearly tabled value of the unknown as 14 1/4 1/97 1/56 1/679 1/776 1/194 1/388 [rearranged in descending order in the bracketed conclusion of Problem 31 of Document IV.1 below].... The second case again opens up two possibilities: either one solves the $(a/b)\ x = C$ by first completing the division of C/a and then multiplying this quotient by b. So it is in [Problem] No. 24, where at first 8 is divided into 19 to obtain 2 1/44 1/8, and then 7 times 2 1/4 1/8 produces 16 1/2 1/8 [the value of the unknown]. Or, [in the second possibility] one divides a/b into 1 and multiplies this quotient by C, as is apparently done in Problems No. 28 and 29 [which see in Document IV.1 below for details].[50]

One of the most interesting critiques of Cantor's views, as well as those of Otto Neugebauer who followed him, was that of T. Eric Peet, as he comments on the solution of Problem No. 26 [as given above but without the bracketed additions I have included]:[51]

> The arithmetical operations here performed [in Problem No. 26] are obvious. The number 4 is taken, its quarter is added to it, giving 5. This 5 is divided into the given 15, and the resulting 3 is multiplied by 4, giving the correct answer 12.

> But what is the thought-process behind this? Cantor thought that it was precisely that of the solution of the equation (1 1/4) x =15; that in Step A 1 1/4 was reduced to an improper fraction 5/4, in Step B the numerator 5 was divided into 15, and in Step C the result multiplied by the denominator 4.
>
> This solution does not fit in with what we know of the handling of fractional quantities by the Egyptians.[52] It seems more probable that the method is that of trial The number 4...is a trial number, chosen because its fourth part involves no fractions. The result of performing the given operation on the trial number is 5. But the given result is 15, and our trial number must, therefore, be multiplied by 3. This is precisely the method which we should follow if told to solve this sum without the use of the algebraic symbol x, and it involves no mathematical principle save that of proportion: if the trial number 4 gives a third of the required result then we must take three times 4.

As Peet goes on to say, there are other problems of this type that are solved directly by division, as for example Problem 30 in Document IV.1. It embraces a common Egyptian phrasing of quantity problems, namely, "If the scribe says to you 'What is the quantity of which 2/3 and 1/10 will make 10?'" The answer, 13 1/23, is calculated by an indirect method that essentially involves the multiplying of 2/3 and 1/10 successively by 1, 4, and 8 to get 13, with 1/30 still to be obtained to produce 10. It is found that the additional multiplier 1/23 is needed to produce the remaining 1/30 and thus obtain the given sum 10. The answer is proved by multiplying the found unknown quantity, i.e., the quotient 13 1/23, by 2/3 and 1/10, to show that these multiplications produce the given number 10.

Problems 28 and 29 in the Rhind Papyrus are supposed by Chace in his translation to have been originally solved by false position like Problems 24-27. But, he believes, the original solutions that would have shown this were not included, and so he reconstructs them (see Document IV.1, notes 27 and 29). Gillings, on the other hand, would classify them as "the very earliest examples of *Think of a number* problems on record."[53] The reader might find this treatment of interest, but I forbear to discuss it here, since Chace's suggestions seem closer to Egyptian procedures.

Sometimes the false unknown assumed was "one", as was so often the case in later algebra. This assumption is made at the beginning of Problem 37 of Document IV.1, as I have already noted earlier when discussing the use of a common denominator in the section on Unit Fractions and the Table of Two. The long mathematical fragment in Berlin Papyrus 6619 (i.e., Document IV.4 below), which solves the problem of dividing 100 square cubits into two squares, one of whose sides is 3/4 of the other, also uses 1 as the false assumption for the side of the first square. So this is comparable to using false position to solve the equation $x^2 + y^2 = 100$ where $y = (3/4)\,x$. As we have already seen in discussing squares and square roots earlier, the sides of the two squares (i.e., the square roots) were determined as 8 and 6. This then is a problem whose solution represents that of a quadratic equation.

The foregoing are some of the principal examples of Aha problems that seem to resemble, in a limited way, algebraic equations and their solutions. And even when worded in terms of practical measures, they seem to represent model problems of arithmetic procedures, achieving a significant step toward general statements that could be used to solve similar problems. Van der Waerden says of the solutions of these Aha-calculations, that they "constitute the climax of Egyptian arithmetic."[54] One can comfortably accept this opinion, while at the same time expressing great doubt (as Peet did) that the key elements of algebra were present in the expression and solution of the Aha problems. And so we can end this discussion of Aha problems much as we began it. The literal symbols that are the essence of algebra with some letters representing given quantities and others representing unknown quantities, and still other standard symbols indicating the arithmetical operations of addition, subtraction, equality, and the practice by rules for operating on polynomials are simply not there,[55] though, as I have just said, the fact that the Egyptian authors seem to be presenting solutions that could be applied to other similar problems is a significant step in the direction of the generality achieved by algebra.

Arithmetic and Geometric Progressions

Problems involving both of the simplest forms of progression were solved by the Egyptians. In a problem of the Kahun Mathematical Fragments (Document IV.3, Cols. 11-12) we find a straightforward procedure for finding a series of 10 numbers in arithmetic progression when the sum is 100 and the common difference of terms is 1/2 1/3. The solution is given in two columns as follows:

	[Col. 11]	[Col. 12]
\ 1	1/3 1/12	100 [items to be divided among] 10 [men] [in arithmetically decreasing amounts]
2	2/3 1/6	13 2/3 1/12
4	1 2/3	12 2/3 1/6 1/12
\ 8	3 1/3	12 1/12
Total	3 2/3 1/12	11 1/6 1/12
		10 1/3 1/12
		9 1/3 1/6 1/12
		8 2/3 1/12
		7 2/3 1/6 1/12
		7 1/12
		6 1/6 1/12.

Thanks to Sylvia Couchoud an important correction has been made in the first line, namely "100[...] 10" items instead of the "110" of the editor Griffith. Hence we do not need the tortuous treatment that Gillings was forced to give the problem (see Document IV.3, n. 1). Instead we see a straightforward application of the arithmetical procedures that match the modern formula for the highest term h, namely $h = (S/n) + (n-1)\ (d/2)$, where S is the sum of the terms (i.e., 100), n is the number of terms (10), and d is the common difference between terms (2/3 + 1/6). Hence Col. 11, starting with $d/2$, i.e., 1/3 + 1/12, shows that when it is multiplied by *n-1*, i.e., 9, the result is 3 2/3 1/12. In moving to Col. 12, *S/n*, i.e., 100/10, becomes 10 and when added to the result of Col. 11, the result gives us the highest term, 13 2/3 1/12. The succeeding terms then are determined by successively subtracting the common difference.

The similar Problem 64 of Document IV.1 is solved in the same way as the Kahun problem. It asks what is the share of each of 10 men if 10 heqat of barley is divided among them in such a way that the excess of barley of each man over his predecessor is *1/8* heqat. The formulation *S/n*, which I used above, is called here the "average share," while *n-1* and *d/2* are given below in the problem as follows: "Take 1 from 10, and the remainder is 9 [as the number of differences, i.e., 1 less than the number of men]. Take 1/2 of the [common] difference, i.e., *1/16* [heqat]. Multiply this by 9 and the result is *1/2 1/16* [heqat]. Add this to the average share [and this becomes 1 *1/2 1/16*, which is the largest share]. Subtract *1/8* heqat for each man until you reach the last one." He then lists each share.

The final problem regarding an arithmetic series that should be discussed is Problem 40 of Document IV.1, which I repeat here:

[Divide] **100 loaves** among 5 men [in such a way that the shares received will be in arithmetical progression and that] 1/7 of [the sum of] the largest three shares is [equal to the sum of] the smallest two. What is the [common] excess [or difference of the shares]?

The procedure is as follows, [if we assume first that] the excess [or difference] is 5 1/2. [Then the amounts that the five men receive are]

23, 17 1/2, 12, 6 1/2, 1; total 60.

[As many times as is necessary to multiply 60 to make 100, so many times must the terms noted above be multiplied to find the correct terms of the series.]

\ 1 60

\ 2/3 40

Total: 1 2/3 100.

[Then] multiply [the above assumed terms] by 1 2/3 [as follows:]

23 it becomes 38 1/3

17 1/2	it becomes	29 1/6
12	it becomes	20
6 1/2	it becomes	10 2/3 1/6
1	it becomes	1 2/3
Total: 60	it becomes	100.

[And so the common excess or difference between any two terms is 9 1/6.]

It can readily be seen that this problem cannot be solved by simply following the procedure of the preceding problems, for it is evident that there will be two unknowns in $h = S/n + (n-1)(d/2)$, namely, h and d. Using algebraic techniques, we would put beside the preceding equation the equation representing the additional condition given in the enunciation of the problem (namely, 1/7 of the sum of the three highest terms is equal to the two lowest terms): $[h + (h-d) + (h-2d)]/7 = (h-3d) + (h-4d)$, and then solve the simultaneous equations for h and d. But this approach was apparently not within the competence of the author. He immediately assumes (using false position) that the constant excess of a term over its predecessor is 5 1/2. We do not know why he took that value. I suspect that he rapidly went through a possible series of 5 terms (those having the lowest term 1) with various trial excesses, first with those consisting of integers and then with those made up of mixed numbers consisting of integers plus 1/2, until he found the one that produced the condition that 1/7 of the sum of the last three highest terms equaled the sum of the two lowest terms. This turned out to be the series 23, 17 1/2, 12, 6 1/2, 1, where the common excess was 5 1/2. He then computed the sum of that series as 60. But one of the given conditions was that the true sum was 100 loaves. Accordingly, he used a proportion to find the true terms, realizing that as many times as it is necessary to multiply the false sum 60 to produce the true sum 100, so many times must each term be multiplied to find the correct terms of the series. He thus found that each term must be multiplied by 1 2/3 to produce the correct terms. As I have suggested by the last bracketed statement, he probably should have subtracted from any one term its predecessor to obtain the true excess (9 1/6). One would have thought that it would have been easier to have produced the true common excess first by multiplying the falsely assumed excess 5 1/2 by 1 2/3. Then after determining the highest term as 38 1/8 he would have simply had to subtract the true common excess 9 1/6 successively four times to produce the remaining terms of the series instead of using his more complicated process of multiplying the remaining terms of the false series by 1 2/3.

We can also note that in determining the terms in the three problems concerned with arithmetic series, the author (or authors) always determined the highest term first and then the successive lower terms by subtracting the common excess.

Now let us move on to geometric progressions. It should be immediately evident to the reader who has read this far that the simplest of all geometric series, namely that produced by doubling, such as 1, 2, 4, 8..., i.e., where the ratio of any term to its predecessor term is 2, lay at the heart of the Egyptian system of multiplication and division, since one could easily make up any intervening whole number multiplier by selectively adding these terms and thereby produce a final product from the addition of the partial products produced by all of the terms making up the total multiplier. We have also seen that another simple geometric series helped to provide unit fractional multipliers: 1/2, 1/4, 1/8, ... This series was composed of the reciprocals of the basic whole number series just described. I note in passing that the first six terms of the unit-fraction series were commonly used for fractions of a heqat, and were represented by special Horus-eye fractional symbols which are pictured in Fig. IV.3 (also consult Document IV.1, note 46).

We can proceed from these two basic geometric series to a more complicated one in Problem 79 of the Rhind Papyrus (Document IV.1). There we see that the Egyptian mathematician was also interested in a geometric progression in which the ratio of any term to its predecessor was 7 and its first term was 7:

[Sum a geometrical progression of five terms of which the first term is 7 and the multiplier of each term is 7.]

A house-inventory [shows how to find the multiplication by 7 to find each term as a product in a series].

[Multiply 2801[56] by 7:]

1	2801
2	5602
4	11204
Total:	19607.

[The same procedure is followed to multiply each term in the following series of five numbers by 7, which then may be summed.]

houses	7
cats	49
mice	343
malt	2401 (*corr. ex* 2301)
heqat	16807
Total:	19607.

The lead column is evidently presented as a shorter way to sum up a series of 5 terms of which the first is 7 and the ratio of each term to its predecessor is also 7. It indicates that if we start with 2801, then double it twice and add the products opposite the multipliers 1, 2, 4, i.e., multiply it by 7 in the Egyptian way, we shall find the sum 19607, which is confirmed as the sum of the specified 5-term series in the second table by the normal method of multiplying each term by 7 in order to find products that add up to the sum of the series, i.e., 19607. The difficulty in this problem is simply that we are not told how 2801 was selected as the number to multiply by 7 in the first table. As Neugebauer pointed out,[57] if we sum a series consisting of 1 + the first 4 terms of the 5-term series under consideration, we would get a total of 2801, which, as the first table of Problem 79 indicates, yields 19607 when multiplied by 7, a product equal to the sum of the 5-term series arrived at in the second table of Problem 79. But this would not be much of a short cut since one still would have to find the sum of the first 4 terms before multiplying it by 7. Gillings suggests that it "is quite possible that 2,801 had merely to be read off from a table...prepared long before...."[58] But, since we have no table of various geometric progressions such as that proposed by Gillings, we have no documentary support for this suggestion.

Pefsu Problems

Pefsu is the conventional term for the strength of bread or beer made from a heqat of grain; it is often translated as "the cooking" or "the cooking ratio (or number)." Historically three forms of the word were used, all originating in the verb *fśi*, 'to cook'. The first was probably *fśw*, the second was *pfśw*, the third was *pśw* (see Document IV.1, nn. 103-04). We can express it in a general mathematical expression as

Pefsu = (no. of loaves of bread or jugs of beer) / (no. of heqats of grain).

Hence the higher the pefsu number, the weaker the bread or beer.

The pefsu number was widely quoted in offering lists like that specified for the Feast of the Rising of Sothis quoted from the Medinet Habu calendar in Volume 2 above (p. 271). Furthermore, it was a particularly useful number to know in a barter economy, as we shall see when we point to the number of problems devoted to the exchange of bread and beer.

In our documents below there are twenty problems concerned with pefsu, 10 in Document IV.1 and 10 in Document IV.2. Let us look at the first of the 10 pefsu problems in the Rhind Papyrus (Problem 69 of Document IV.1), which determines the pefsu of 80 loaves of bread made from 3 1/2 heqat of meal:

3 1/2 heqat of meal is made into 80 loaves of bread. Make known to me the amount of meal in each loaf and their pefsu (*pfśw*) [i.e., cooking potency].

Multiply 3 1/2 so as to get 80.

1	3 1/2
10	35
\20	70
\2	7

\ 2/3	2 1/3
\ 1/21	1/6
\ 1/7	1/2.

The pefsu is 22 2/3 1/7 1/21.

[Proof:]

\ 1	22 2/3 1/7 1/21
\ 2	45 1/3 1/4 1/14 1/28 1/42
\ 1/2	11 1/3 1/14 1/42
[Total :	80].

[3 1/2 heqat makes 1120 ro, for]

\ 1	320
\ 2	640
\ 1/2	160
Total:	1120 ro.

[Hence] multiply 80 so as to get 1120.
The procedure is as follows:

1	80
\ 10	800
2	160
\ 4	320
Total: [14]	1120.

So the amount of meal in one loaf [is 14 ro or] *1/32* heqat 4 ro.
[Proof, with the Horus-eye fractions given here in Italics:]

1	*1/32* [heqat] 4 ro
2	*1/16 1/64* [heqat] 3 ro
4	*1/8 1/32 1/64* [heqat] 1 ro
8	*1/4 1/16 1/32* [heqat] 2 ro
\ 16	*1/2 1/8 1/16* [heqat] 4 ro
32	1 *1/4 1/8 1/64* [heqat] 3 ro
\ 64	2 *1/2 1/4 1/32 1/64* [heqat] 1 ro

The result is 3 1/2 heqat of meal [for the 80 loaves, as was specified].

The first table evidently uses the pefsu ratio to find the pefsu. In the Egyptian way, this is to find the total multiplier of 3

1/2 that will yield 80. As the checks in the multiplier column show, the partial multipliers that produce the partial products that sum at 80 are the ones that add up to 22 2/3 1/7 1/21, which is the pefsu. Then there follows a table verifying the preceding division by multiplying 22 2/3 1/7 1/21 by 3 1/2 to get 80. In the next table the scribe converts the 3 1/2 heqat of meal into ro by multiplying 320 (the number of ro in 1 heqat) by 3 1/2. The result is 1120 ro. Then since there are 80 loaves, the scribe must then multiply 80 so as to get 1120. The result is 14 and thus the amount of meal in 1 loaf is 14 ro, i.e., *1/32* heqat and 4 ro of meal [since *1/64* heqat = 5 ro]. Finally a calculating proof is given to show that the multiplication of *1/32* heqat and 4 ro by 80 does indeed result in 3 1/2 heqat of meal producing the 80 loaves. Problem 70, which follows the problem we have just analyzed, is the same kind of problem, as the reader will readily see. Another problem of this sort is Problem 20 of Document IV.2 and it need not be repeated here since it uses a bread of given pefsu and a given number of loaves of bread to find the quantity of grain. This allows us to move on to a somewhat different problem concerning the pefsu of a diluted beer, namely, Problem 71 of Document IV.1.

From 1 **des-jug of beer** 1/4 has been poured off and then the jug has been refilled with water. What is the pefsu of the diluted beer?

Calculate the amount of besha (i.e., a kind of grain mixed with fruit?) in 1 des of beer; the result is *1/2* [heqat] of besha. Take away 1/4 of it, namely, *1/8* [heqat]. The remainder is *1/4 1/8* [heqat]. Multiply *1/4 1/8* [heqat] so as to get 1 [heqat]. The result is 2 2/3, which is the pefsu [of the diluted beer].

It is evident that the pefsu of the original beer results from the division of 1 by 1/2, namely, 2. Now if 1/4 of the beer is drawn from the des-jug and the empty 1/4 of the des-jug is refilled with water, the amount of besha has been reduced by 1/8, leaving *1/4 1/8* heqat (3/8 heqat in modern notation). Thus the pefsu of the diluted beer is the result of the division of 1 by 1/4 1/8, or, as the scribe says, by multiplying *1/4 1/8* so as to obtain 1.

Some sort of similar diluted beer is found in 8 of the 10 pefsu problems in Document IV.2. It seems to be designated "1/2 1/4 malt-date beer" (see Problems 5, 8, 9, 12, 13, 16, 22, and 24); but why this diluted malt-date beer is so designated is not clear (see Document IV.2, note 7). It usually seems to be made with a strength of pefsu 4. It is ordinarily compared to a stronger beer (and thus one having a lower pefsu number) which is specified as made with Upper Egyptian Grain and has a pefsu of 2. Problem 15 of this document is a simple one for finding the quantity of the stronger beer made from Upper Egyptian Grain when the pefsu is given as 2 and the quantity of grain is specified as 10 heqat, yielding, of course, 10 1-des-jugs.

The next type of pefsu problems that demands attention concerns the exchanging of loaves of bread of differing pefsu or the exchanging of bread and beer. Problems 72-78 of Document IV.1 are all of that type. A simple example of the exchanging of one set of loaves of bread having one pefsu with another set of loaves with a different pefsu is Problem 73:

If it is said to you, "100 loaves of [pefsu] 10 are to be exchanged for loaves of [pefsu] 15. How many of the latter will there be?"

Calculate the amount of wedyet-flour in these 100 loaves; it is [10] heqat. Multiply 10 by 15. This is 150. Reply [then] that this is [the number of loaves for] the exchange.

The procedure is as follows: 100 loaves of [pefsu] 10 would be exchanged with 150 loaves of [pefsu] 15. [It takes] 10 heqat.

The solution is by use of a simple 4-term proportion. Since the amount of grain producing the two sets of loaves remains the same (i.e., 10 heqat), the ratio of the number of the loaves (100) to pefsu (10) of the first set must be the same as that of the unknown number of loaves to pefsu (15) of the second set. Hence, as many times as 10 must be multiplied to produce 100, namely 10, that many times must 15 be multiplied to produce the number of loaves of the second set. Hence the answer is 150 loaves. Problem 75 is solved in precisely the same way and Problem 74 of the same document is only slightly different. Instead of exchanging the whole 1000 loaves of bread with pefsu 5 for a single set of loaves of bread with another pefsu, the scribe exchanges 1/2 for a bread with pefsu 20 and 1/2 into a bread with pefsu 30. So he treats the two exchanges as two simple problems to be solved in the conventional way I have already outlined.

Problem 72 could also have been solved in precisely the same way; but, though the scribe clearly was still using the basic pefsu proportion, the actual solution was different. Its procedures reflect rearrangement of the proportion by what would be called alternation and subtraction in modern terms. First let me present the problem as given:

Example of exchanging loaves for other loaves. You are told that there are 100 loaves of [pefsu] 10 to be exchanged for some number of loaves with [pefsu] 45. [How many of these will there be?]

Calculate the excess of 45 over 10; it is 35. Multiply 10 so as to get 35; it is 3 1/2. Multiply 100 by 3 1/2; it is 350. Add 100 to it; it is 450. Say then that 100 loaves of [pefsu] 10 are exchanged for 450 loaves of [pefsu] 45, making in wedyet-flour 10 heqat.

The original proportion, which we might suppose he thought of as 100/10 = unknown no./45, was apparently transformed into 45/10 = unknown/100, by interchanging 45 and 100 in the original proportion, for then he approximates the further procedure of transforming the alternated proportion by subtraction: that is, he subtracts 10 from 45, obtaining 35, which he divides by 10 (i.e., using normal Egyptian procedures, he multiplies 10 so as to get 35), the quotient thus being found as 3 1/2. He then multiplies 100 by 3 1/2, getting 350. To the latter he adds 100 to find the correct answer of 450 loaves. We can see this is equivalent to taking the arithmetical steps dictated by the proportion (45-10)/10 = (unknown -100)/100, by completing the division of the left side, and finding it to be 3 1/2, then multiplying that by the fourth term from the right side (100), to get 350 on the left side, thus leaving the term (unknown - 100) on the right side, and then finally adding the negative term (100) to the left side to get 450, which thus equals the unknown number of loaves obtained by the author. This whole procedure certainly shows how cleverly the scribe could manipulate proportions.[59] I have purposely stated these terms in numbers plus the word unknown so as not to prejudice the reader to think exclusively in terms of algebraic symbolism.

In treating Problem 74 we saw how the scribe accomplished the exchange of 1000 loaves of pefsu 5 produced from 200 heqat of flour for two sets of loaves, each produced from 100 heqat of flour, the first having a pefsu of 10 and the second a pefsu of 20. This was done by separating the problem into two simple problems. Now Problem 76 of the same document at first glance seems similar. However, the objective here is not to find the differing numbers of loaves in each set derived from the same amount of flour but rather to exchange 1000 loaves with pefsu 10 and thus made from 100 heqat of flour for 2 sets of bread of differing pefsu but having the same number of loaves, one with pefsu 20 and the other with pefsu 30, a very much more difficult problem for him. Algebraically we can write the simple equation $1000/10 = (1/20 + 1/30)x$, i.e., $(x/12)=100$, and so the solution is immediately evident as 1200, the number of loaves in each set. But here is the solution given by the scribe:

[One loaf of each kind will take] 1/20 and 1/30 [of a heqat]. [As parts of 30] 1/20 is 1 1/2 and 1/30 is 1. [Added,] the total is 2 1/2.

Multiply 2 1/2 so as to get 30:

1	2 1/2
\ 10	25
\ 2	5

Total: 12.

[Therefore 2 1/2 is 1/12 of 30, so that 1/20 1/30 equals 1/12. Two loaves, one of each kind, will take 1/12 of a heqat and 1 heqat will make 12 loaves of each kind.]

The quantity of wedyet-flour in the 1000 loaves is 100 heqat. Multiply 100 by 12; the result is 1200, which is [the number of loaves of each kind, i.e., for loaves of pefsu] 10 [and those of pefsu] 20. [In summary,]

1000 loaves of [pefsu] 10, making 100 heqat of wedyet-flour can be exchanged for

1200 loaves of [pefsu] 20, making 1/2 of 100 [heqat] and 10 [heqat, totaling 60 heqat of wedyet-flour], and

1200 loaves of [pefsu] 30, making 1/4 of 100 [heqat] and 15 [heqat, totaling 40 heqat of wedyet-flour].

Thus the scribe's first step was to assume that there was 1 loaf of each kind of bread. There would thus take 1/20 + 1/30 heqat to produce these loaves, since the pefsu are respectively 20 and 30. He then clears the fractions by assuming, in the Egyptian way, 30 as a common denominator (as we have seen done before when the scribe used red auxiliaries). Hence as parts of 30, he correctly finds 1 1/2 and 1 respectively, or 2 1/2. Then he multiplies 2 1/2 so as to obtain 30. The answer is 12, and the reciprocal of that is 1/12. Hence 2 loaves, one of each kind, will take 1/12 of a heqat and 1 heqat would make 12 loaves of each kind of bread. But 100 heqat of flour produced the 1000 loaves. Thus if 100 is multiplied by 12 we get 1200, the number of each set of loaves, as the author correctly indicates in his concluding summary.

Another problem that seems to follow the reverse of the procedure of Problem 76 is Problem 21 of Document IV.2. But, in fact, this is not so. Let us first quote the actual statement of the problems and its solution:

[Col. XXXVIII]

[Lin. 1] Example of calculating the mixing (*š*ʿ*bn* = *šbn*) of offering-bread.

[Lin. 2] If someone says to you: "20 measured (?) [as Horus-eye fraction] ⁓ (i.e., *1/8* heqat of grain) and 40 measured (?) as [Horus-eye fraction] ᐅ (i.e., *1/16* heqat of grain)."

[Lin. 3] You are to take 1/8 of 20; because the [Horus-eye sign] ⁓ is 1/8.

[Lin. 4] The result is 2 1/2. You are to take 1/16 of 40 because

[Lin. 5] [the Horus-eye sign]  is 1/16. The result is 2 1/2. You are to calculate
[Lin. 6] the total of these [fractions of 20 and 40]. The result is 5. You are to calculate the total
[Col. XXXIX]
[Lin. 1] of these [initial numbers 20 and 40]. The result is 60. Then you divide 5 by 60, and
[Lin. 2] the result is 1/12 (*corr. ex* 1/16). Behold, the mixture is 1/12 (*corr. ex* 1/16). You will find [that it is] correct.

It is by no means a clear text and the twice repeated fraction 1/16 of the last line has been corrected to 1/12. Peet's interpretation of this problem makes sense and I now give it (with the observation that he loosely translates "heqat" as "gallon").[60]

> M[oscow. Problem]21 shows how to find the "average" of two lots of loaves, 20 containing each 1/8 gallon of flour, and 40 containing each 1/16 gallon. The 20 are shown to contain in all 2 1/2 gallons, and the 40 also 2 1/2 gallons. The total flour is 5 gallons, and this, when made into 60 loaves (20 + 40), will allow 1/12 gallon for each loaf. It is only when we reach the answer that we fully understand what was meant by the "average"; the total number of loaves is to remain the same, but all the loaves are to be of the same size.

One should see, in connection with this interpretation, Document IV.2, note 32, for Gillings' suggestion of its possible relation to the calculating of a harmonic mean, since 1/12 is the harmonic mean between 1/8 and 1/16.

The final two pefsu problems in Document IV.1, nos. 77 and 78, involve respectively exchanging bread for beer and beer for bread. They use the simple pefsu ratio in the solutions, as we can illustrate by repeating Problem 78 here:

Example of exchanging bread for beer. If it is said to you: "100 loaves of bread of [pefsu] 10 are to be exchanged for a quantity of beer of [pefsu] 2, [reason as follows to find the quantity of beer]."

Reckon the amount of wedyet-flour in 100 loaves of [pefsu] 10; it is 10 [heqat]. Multiply 10 by 2; it makes 20. Say then that this [i.e., 20 des] is [the amount of beer it takes for] the exchange.

We can briefly note here that the distribution of bread and beer portions played a significant part in the domestic economy of the priests and scribes of the Temple of Illahun. Ludwig Borchardt published, translated, and wrongly analyzed a table of the distribution of 70 loaves of bread, 35 jugs of *sḏ3*-beer, and 115 1/2 jugs of *ḥpnw*-beer in 41 2/3 portions to various priests, scribes, and officials of the temple at Illahun[61] (see Figs. IV.33-34 below; the 41 2/3 being the addition arrived at by multiplying each pair of items of the first two columns which are under the "Teile?" heading). Hence, each portion of bread (determined by dividing 70 by 41 2/3) was approximated by the accountant as 1 2/3 for 1 2/3 1/75, that of the first beer (determined by the division of 35 by 41 2/3) was approximated as 2/3 1/6, i.e., 1/2 of 1 2/3, for 2/3 1/6 1/150, while that of the second beer (determined by the division of 115 1/2 by 41 2/3) was approximated by 2 2/3 1/10, for 2 2/3 1/10 1/250 1/750.

Ancient Egyptian Geometry: Areas

It is evident from the earlier sections of this chapter on measurement that calculations regarding areas of land were among the earliest and most widespread of mathematical activities. Peet is correct in his judgment quoted over note three in the first section, that the tomb of Metjen, of the late third and early fourth dynasty, shows that the early Egyptians had knowledge of the correct determination of the area of a rectangle of land, e.g., where the tomb text says that "there has been conveyed to him (Metjen) in return for compensation 200 arouras of arable land by many of the king's people...[and] a house 200 cubits long and 200 cubits wide," which is an obvious reference to its rectangular, indeed square, floor plan (see Vol. 1, p. 152). By the time of the mathematical papyri of the Middle Kingdom, we find stated the express formula: Area = base x altitude. A good example of this formulation is found in Problem 6 of Document IV.2, which I insert here along with the comments I have added to that problem in the document below:

[Lin. 1] **Example of calculating a rectangle.**
[Lin. 2] If someone says to you: "A rectangle 12 setjat [in area] [has] a breadth 1/2 1/4 of its length. [Calculate its area.]"
[Lin. 3] Calculate 1/2 1/4 to get 1. The result is 1 1/3.
[Lin. 4] Take this 12 setjat 1 1/3 times. The result is 16.
[Lin. 5] Calculate its square root. The result is 4 for its length; [and] 1/2 1/4 of it is 3 for
[Lin. 6] the breadth. The correct procedure is as follows [see the rectangle in line 6 of column VIII in Fig. IV.6c, marked with the area of 12 in the center, the length of 4 above, and the breadth of 3 on the left side. The figure illustrates the problem and represents a kind of proof. Then follows the calculation of the area, which shows that 3 x 4 does indeed equal 12, the specified area]:
\1 4
\2 8
[Total: 12].

As I suggest in my bracketed addition to the problem, the diagram with breadth, length, and area all marked with numbers is an important addition to the solution. Indeed it gives to the result a kind of generality as a model for all such rectangular areas, i.e., it reinforces the generality of the first line. The reader should also consult Problem 49 of Document IV.1 and its note 66.

There is no evidence of the use of a formula for a triangle in the early tombs of the third millennium, but when it does appear in the mathematical texts of the Middle Kingdom, it expresses the area of the triangle in terms of a rectangle one side of which is equal to half the base of the triangle and the adjacent side equal to the triangle's height, or so it seems to most students of Egyptian mathematics. Let us look first at Problem 51 of Document IV.1:

Example of producing (i.e., calculating) [the area of] a triangle *(spdt)* of land. If it is said to you: "What is the area of a triangle of 10 khet on the *mryt* (most likely, the 'height' or 'kathete'; less likely, the 'side') of it and 4 khet on the base of it?"

The Procedure is as follows:

1	400 [cubits, i.e., 4 khet]
1/2	200 [cubits, i.e., 2 khet]

1	1000 [cubits, i.e., 10 khet]
2	2000.

Its area is 20 setjat.

Take 1/2 of 4, namely, 2, in order to get [one side of] its [equivalent] rectangle. Multiply 10 [the other side of the rectangle] times 2; this is its area [i.e., the area of the rectangle and thus of the triangle].

Since the Rhind Papyrus was the first Egyptian mathematical tract published and examined in detail, its above-quoted Problem 51 was the first Egyptian treatment of the area of a triangle studied by historians of Egyptian mathematics. It was first believed that the triangle as drawn (see Fig. IV.2kk, Plate 73) was a scalene triangle but almost an isosceles triangle (which it was meant to be but not drawn carefully), with one of the almost equal sides marked on its outside as 10 khet and the base labeled on its outside as 4 khet, these two quantities being designated respectively in the text as the *mryt* and the *tpr* and translated as "side" and "base." Hence it was thought that the area of the triangle was approximated as the product of one-half its base and a side. Two scholars to hold this view were the great editor Chace and, more recently, the subtle critic Michel Guillemot (see Document IV.1, endnote 68). But, as I say in that same note, I and many others (e.g., Gunn, Peet, Struve, Gillings, and Couchoud) believe that the *mryt* should be translated in a mathematical context, as "height" or at least "kathete" (the "perpendicular" from the base to the apex of the triangle opposite the base). Gunn, and later Peet as well, gave a satisfactory philological explanation for rendering *mryt* as "height" or "kathete" in its alternate meaning as "quay" or a horizontal structure erected over the sloping bank at the river's edge (again see the endnote just cited). The same problem with the same quantities expressed appears as Problem 4 in Document IV.2. In both cases, it seems to me, the area of the triangle is expressed as a rectangle whose sides are equal respectively to half-the-base and to the altitude.

Even more convincing evidence that the correct formulation for the area of a triangle is understand by the Egyptians is found in Problem 7 of Document IV.2:

[Lin. 1] **Example of calculating a triangle.**
[Lin. 2] If someone says to you: "[There is] a triangle with area of 20 [setjat] and 'bank' (*idb*, i.e., the ratio of height to base) of 2 1/2."
[Lin. 3] Double the area. The result is 40. Take it 2 1/2 times.
[Lin. 4] The result is 100. Take the square root; the result is 10. Call up 1 from 2 1/2.
[Lin. 5] The result is 1/3 1/15. Apply this to 10. The result is 4.
[Lin. 6] [Hence it is] 10 [khet] in the length (i.e., kathete) and 4 [khet] in its breadth.

The problem is to find the height and the base of a triangle when the ratio of these two quantities is 2 1/2 : 1 and the area is 20. As I have noted in Document IV.2, endnote 11, the solution of this is similar in modern terms to solving two simultaneous equations: (1) Area = 1/2 base x altitude, and (2) height = 2 1/2 x base. The first move is to double the area, which yields 40. Then we multiply 40 by 2 1/2, yielding 100. Then the square root of 100 is computed as 10. Now if we get 1 by finding a multiplier of 2 1/2, that multiplier must be 1/3 1/15 (6/15 in modern parlance), and if instead of 1 we get 10, as the conditions of the problem demand, then we must multiply 1/3 1/15 by 10 and the answer is 4, the length of the base, and so the height is 2 1/3 times the base, or 10.

Hence it is now obvious that the key step bearing on our discussion of determining the scribe's formula for the area of the triangle was to double the given area of the triangle by constructing on the base a rectangle whose adjacent sides had to be the height and the base of the triangle. Thus it is obvious to us graphically that the area of the triangle underlying the solution of this problem has to be one half of the base times the height (here called "length" instead of "*mryt*").[62] I believe that the height was called "length" in this and the similar Problem 17 of Document IV.2 not because it was a *right triangle* tipped on its side, as Struve, Neugebauer, and Guillemot held, but simply because it was tipped on its side, thus making the kathete the longer measurement. Actually the triangle is only seen clearly to be so tipped on its side in the text of Problem 17 (see the hieratic text in Fig. IV.6m). In this problem, however, it was so crudely drawn that it cannot be affirmed to be a right triangle. The height or kathete was perhaps also called the "length" in these two problems because in that tipped position the longer side of the equivalent rectangle was probably also called its "length."

Hence some such simple graphic observations were in all likelihood behind the discovery of the basic formula for the area of the triangle. For if any scalene triangle ABC (see Fig. IV.4c) is divided into two right triangles ABD and ADC by drawing a perpendicular AD from the apex to the base and each triangle after that division is then doubled by constructing right triangle AEB equal to triangle ABD on the left and right triangle AFC equal to triangle ADC on the right so that each pair shares a common hypotenuse, then the total rectangle BEFC formed by the four right triangles is indeed double the initial triangle ABC. And so that initial triangle would have had to be one half the final rectangle, i.e., half the product of the latter's base BC times its height BH (= AD, the height of the triangle ADC). If one tried to use either side AB or AC to construct a rectangle on the base BC, it would be obvious by inspection that the resulting rectangle would be more than double triangle ABC since AB or AC would be longer than the height AD. As is evident from my comments in the preceding paragraph, the reader should also read carefully Problem 17 of Document IV.2. Needless to say, if Problems 7 and 17 concern a right triangle, as the editor of the Moscow Papyrus (Struve) believed (see the figures of the triangle given in the hieroglyphic transcriptions of Problems 7 and 17 in Figs. IV.6c and IV.6m), the "length" or *myrt* would simultaneously be the "height" and a "side" of the initial triangle and the correct formula for the area would be immediately evident in the problem's solution, and thus the preceding analysis would not be relevant to Problems 7 and 17. It would, however, be completely relevant to Problem 51 of Document IV.1, whether that triangle was meant to be isosceles or scalene.

The question of the meaning of the word *mryt* arises once more in Problem 52 of Document IV.1, which yields the formula for a truncated triangle, i.e., a trapezoid, which I quote from the text below:

Example of a truncated triangle (i.e., a trapezoid). If it is said to you: "What is the area of a truncated triangle of land of 20 khet in its height [or, side?], 6 khet in its base, 4 khet in its truncating line?"

Add its base to its truncating line; it makes 10. Take 1/2 of 10, i.e., 5, in order to get [one side of] its [equivalent] rectangle. Multiply 20 [the other side of its rectangle] times 5; it makes 10 (10 ten-setjat). This is the area.

The procedure is as follows:

1	1000 [cubits, i.e., 10 khet]
1/2	500
\ 1	2000
2	4000
\ 4	8000
Total:	10,000 [cubit-strips, as in the preceding problem].

Its area is 100 setjat (10 ten-setjat).

If we assume, as I have assumed in the translation and have concluded in the previously discussed triangle problems, that *mryt* is "height", then the calculations given yield the correct formula for the truncated triangle, namely A = [1/2 (truncating line + base line) x height]. In this case we would certainly reject the alternate translation of *mryt* as "side". Peet's comments concerning this problem reinforce what I have said:[63]

> Here [in Problem 52] we halve the sum of the parallel sides "in order to get its rectangle" and multiply by the *meryet*. Surely the idea of halving the sum of thr parallel sides can only have come from a graphic solution such as that shown in Fig. 4 [*Author* = my Fig. IV.35b], and with the truth so clearly in front of their eyes it is not possible to believe that the Egyptians were so silly as to multiply half this sum not by the correct vertical height of the figure but by one of the slant sides (assumed in this case to be equal),... and, if here, so also in the case of the triangle.

I forbear to discuss here Problem 53 of Document IV.1 concerning an area composed of a triangle and a trapezoid because of the chaotic state of the text, but the reader will find some effort to straighten it out in the translation of the document below.

The last area problems needing discussion are those concerned with the area of a circle, in many ways the most interesting area problems solved by the Egyptians. Their interest lies in the fact that the basic procedure involved quadrature, that is, finding a rectangular figure (in fact, a square) equal to a circle. Thus the Egyptians' procedure of reducing circles to squares stands at the head of a long line of efforts of increasing sophistication to find the areas and volumes of figures bounded by curved lines or curved surfaces in terms of figures bounded by straight lines or plane surfaces. While not achieving the elegance of the Greeks (and above all of Archimedes) their solution demands attention. This oft-praised solution of the circle equated the area of a circle of diameter 9 with that of a square of side 8. To generalize it we can say that it involved finding the area of a circle by subtracting 1/9 of its diameter from the diameter and then squaring the remainder, which, as has often been pointed out, achieves an approximation 256/81 (3.1605) for the modern symbol π as compared to the modern approximation 3.1416. To put it in another way, as Gillings does, the Egyptian solution of the area of a circle of diameter 9 khet, i.e., the square of 8 khet, was 64 setjat, while, if we used the modern approximation for π, the area would be approximately 63.6174 setjat, "so that the Egyptian value is in error by less than 0.6 of one percent."[64] Let me now give as an example of the Egyptian determination that is found in Problem 50 of Document IV.1:

Example of producing [the area of] a round field of diameter of 9 khet. What is the reckoning (*lit.*, *rḫt*, knowledge) of its area (*3ḥt*)?

Take away 1/9 of it (the diameter), namely 1; the remainder is 8. Multiply 8 times 8; it makes 64. [Therefore,] the amount of it in area is 64 setjat.

The procedure is as follows:

1	9
1/9	1;

this taken away leaves 8.

1	8
2	16
4	32
\8	64.

The amount of it in area is 64 setjat.

As Gillings notes,[65] the assumption by the author of a diameter of 9 khet was "a matter of arithmetical convenience and not because it is a really practical problem," since it results in an area of over 40 acres and a "circumference of nearly a mile." The procedure given here is also found in the course of Problems 41 and 42 of Document IV.1 and of Problem 10 of Document IV.2, all devoted to volumes. The principal question raised by this good approximation is: How did it arise? Problem 48 of Document IV.1 may give us the answer:

[Compare the area of a circle (or, better, an octagon equal to it?) and its circumscribing square.]

[Circle of diameter 9 (or, better, an octagon = sq. of side 8)?]		[Square of side 9]	
1	8 setjat	\1	9 setjat
2	16 setjat	2	18 setjat *[cont.]*

4	32 setjat	4	36 setjat
\ 8	64 setjat	\ 8	72 setjat
[Total:	64 setjat]	Total:	81 setjat.

The bracketed additions (except for the alternative statements which I have added in parentheses) are those added (without brackets) by Chace to the two tables found in the hieratic text. Thus he believed that the two tables which appeared as Problem 48 were simply a comparison of the calculations yielding the areas in setjat of a circle of diameter 9 and the square of side 9 in which it can be inscribed. But the difficulty of this explanation is that the figure accompanying the tables shows a square with an inscribed octagon and not with an inscribed circle. This octagon was crudely drawn to be sure, thus producing unequal corners in the square (see below Fig. IV.2ii, Problem 48), but it was possible, as Vogel proposes, that it was meant to be a symmetrical and semiregular octagon created by trisecting the sides of a square of side 9 and cutting off the four triangular corners each equal to 9/2 (see Fig. IV.36 below). But whatever the structure of the octagon, I believe it probable that the author assumed in Problem 48 that it was the inscribed octagon, itself being approximately equal to the circle of diameter 9, that was being compared to the square of side 9. I remark in note 64 of Problem 48 that this was the suggestion of Vogel and later of Gillings. I briefly describe there Vogel's proposal for a possible method, on the part of the scribe, to show that the inscribed octagon and a square of side 8 are approximately equal (63 setjat and 64 setjat; and see Fig. IV.36). But there is no trace of this calculation in the Egyptian text (except for the poorly drawn octagon inscribed in the square). Nor is there any trace of Gillings' suggested graphic method of assuming the approximate equality of the octagon and square of side 8, but I shall give his rather more detailed proposal here:[66]

> By drawing a diagram as that shown in Fig. 13.6 [=Fig. IV.37 below, upper left] on a piece of papyrus, the scribe would conclude that the octagon was pretty closely equal in area to the inscribed circle because some portions of the

circle are outside the octagon and some portions of the octagon are outside the circle, and mere observations by the naked eye suggest these are roughly equal. He then would sketch a square of sides representing 9 khet, trisect the sides, join the adjacent points of division, and then by drawing all the lines necessary to actually see or visualize each of the square khets or setats he can count these squares in any way he pleases to find the number of them in the octagon (see Figure 13.6 [*Author* = my Fig. IV.37, upper right]).

Now the Egyptian scribes found the areas of squares and rectangles with ease. Then if the two top shaded corners of 4 1/2 setats (or square khets) each, which add to 9 setats, were to replace the top row of 9 setats and if, similarly, the two bottom shaded corners of 9 setats were to replace the left-hand column of 9 setats, then the figure remaining [see Fig. IV.37, lower figure], would be a square instead of an octagon, the area of which the scribe can easily calculate.

The scribe could now properly conclude that the area of the circle inscribed in a square of side 9 khets is very closely equal to the area of a square of side 8 khets.

Therefore, in Problem 48 the scribe finds the total number of setats in this square of eight rows of 8 setats, which he gives as the area of a circle of diameter 9 khets. Of course he certainly knows his method is not *exact*, because he has, so to speak, cut off one setat twice—the one in the top left-hand corner. But his method allows him to find a square nearly equal to a circle, so that we can, "*en caprice*," as it were, credit A^{c}h-mosè with being the first authentic circle-squarer in recorded history!....

Nowhere in this problem does the scribe give the direction, "Take away thou one-ninth of the diameter," as he does in the four other Problems 41, 42, 43, and 50, for this is where he is showing how he discovered his now classical rule.

I hardly need point out to the reader that, though the scribe may be hinting to the reader the nature of his graphical solution by specifying that the quantities being multiplied in both tables are setjat, i.e., collections of unit square khet, he is certainly not explicitly "showing how he discovered his now classical rule." None of the three diagrams in Fig. IV.37 is given in the papyrus or anywhere but in Gillings' fertile imagination; still a later drawing of an ellipse and an approximately equal rectangle at the temple of Luxor (see Fig. IV.42) gives some evidence of a graphic procedure like that suggested by Gillings for the quadrature of the circle (compare Borchardt's suggestions in note 67, and particularly Couchoud's brief analysis there). The only thing we can be fairly sure of is that the author of the Rhind Papyrus does assume in Problem 48, for some unspecified reason, that the inscribed octagon is equal to a square of side 8 and is as well equal to the inscribed circle of diameter 9, which then becomes a geometrical illustration of his classical rule for the measurement of a circle.

But Michel Guillemot, in the article cited in note 66, rejects the assumption made by Vogel and Gillings that the octagon intended in Problem 48 is a symmetrical and semi-regular octagon of the form they suggest (again see note 66). He proposes instead that the inscribed octagon as figured in that Problem was drawn by cutting off, from the circumscribed square of side 9, two diagonally opposite corner triangles each equal to 9/2 and the two other corner triangles each equal to 8/2 (see Fig. IV.40). Such a proposed octagon was in fact much closer to the octagon drawn in the papyrus, and furthermore its area is actually 64 rather than some approximation thereof, as was the case in the former explanations of Vogel and Gillings. Even more important, he suggests that Problem 48 was not an effort to show how the Egyptians geometrically discovered their rule that the area of a circle of diameter 9 was approximately equivalent to that of a square of side 8 but rather was an attempt to show that the previously discovered rule satisfied Egyptian knowledge of the geometry of the areas of squares and rectangles. As for the prior discovery of the quadrature rule for the circle, Guillemot speculates further that the previously discovered quadrature rule was discovered through a calculating approximation arising from the economic necessity of measuring the volumetric content of a cylindrical granary. Such a volume was found always by multiplying the area of the circular disk or cover by the granary's height. That approximation of the square equal to the disk (i.e., a square with side 8 2/3 1/6 1/18, as determined in Problem 42), he claims, was one too excellent to have been obtained by a "heuristic geometry," or as the earlier authors I have quoted would say, by a "graphic procedure." Guillemot's argument and conclusions about the rule's discovery (see Fig. IV.41 below) are presented with his usual care and imagination.

Incidentally, it is of interest that, of the four problems in Document IV.1 giving the calculation of the area of a circle, three use the diameter of 9 (perhaps because of its ease of trisection, as Gillings notes in his suggested graphic solution), while the fourth (Problem 42) gives a circle of diameter 10. But in all four cases the procedure is so generalized that the first step is always specified as taking 1/9 of the diameter. (In fact it was even mistakenly added to Problem 43, as is suggested by Gillings' version, the third of the versions of this proposition which I have given in Document IV.1 below.) So it is evident that the area of a circle is one more case (like those of the rectangle and triangle) where the scribe gave specific illustrative problems, all of which were exemplifying a general rule of calculating a specific kind of area.

In concluding our section on the quadrature of a circle, we should note that there is some evidence of a similar interest by ancient Egyptians in the quadrature of an ellipse, as the result of the discovery by Borchardt of an ellipse and an approximately equal rectangle scratched on a wall of the temple at Luxor (Fig. IV.42), although it is not certainly known whether the rectangle was given as an aid to constructing the ellipse or as an effort to calculate its area, and furthermore it is later (at least after the time of Ramesses III), and perhaps much later, than the texts we are considering.[67] Still there is no doubt that the use of ordinate measures (i.e., simple vertical straight lines) to help represent or to give the proper form of arcs is evident in architectural sketches. One third-dynasty sketch on a piece of limestone found in Saqqara, presumably from a builder, fixes the form of an arc by designating on successive uniformly spaced vertical ordinates their continually varying heights in cubits, palms, and fingers (see Fig. IV.43).[68]

Volumes

The simplest kind of volumetric formula found in the mathematical papyri was that for a cubical bin, one used for the storing of grain, as is evident in Problem 44 of Document IV.1, where the length, the breadth, and the height are each specified as follows:

Example of reckoning [the volume of] a rectangular granary, its length being 10, its breadth 10, and its height 10. What is the amount of grain that goes into it?
Multiply 10 times 10; it makes 100. Multiply 100 times 10; it makes 1000. Take 1/2 of 1000, namely 500, [and add it to 1000;] it makes 1500, its contents in khar. Take 1/20 of 1500; it makes 75, its contents in quadruple heqat, namely, **7500 heqat of grain.**

The reader will readily see that the scribe's calculations follow the formula $V = l \times b \times h$, each of which is given as 10 [cubits]. Actually, what is being determined first is the number of cubic cubits in the granary by the above formula; then is found the number of khar (this being calculated by multiplying the number of cubic cubits by 1 1/2, since the khar is 2/3 the cubic cubit); then finally the number of khar is converted into the so-called quadruple heqat by taking 1/20 of the khar; and finally multiplying the number of quadruple heqat by 100. In this way the author shows how to find the number of heqat of grain that can be installed in the cubical granary. The following problem (No. 45) gives the number of heqat of grain filling a cubical bin and determines the dimensions of the bin. I am not interested at this point in the grain measurements themselves since I have already spoken of the fractions and multiples of heqat used in granary problems in the earlier section on measurements in this chapter and since I shall give further information in the notes to the various granary problems in Document IV.1 below. I have also noted earlier that the basic formula for a rectangular volume was also everywhere applied in Document IV.6 below and I need not now present my later treatment of it that is found in the course of giving that document. Hence the Egyptian determination of the volume of a parallelpiped was probably their initial and most fundamental contribution in the field of solid geometry.

Though I have presented the rectangular faced volumes first since rectangular formulas are probably earlier than those involving circular measurements, I must point out that in Document IV.1 the volumes of cylindrical granaries appear in Problems 41-43 and thus before those concerning cubic rectangular granaries in Problems 44 and 45. Let us look at the first part of Problem 41, which describes the calculation of the volume of cylindrical granary succinctly:

> **Example of making (i.e., calculating the volume of a) round (i.e., cylindrical) granary of [diameter]** 9 and [height] 10.
> Take away 1/9 of 9, namely, 1; the remainder is 8. Multiply 8 times 8; it makes 64. Multiply 64 times 10; it makes 640 [cubic] cubits. Add 1/2 of it to it; it makes 960: **the calculation of [the content of] it in khar** (*ḫ3rw*). Take 1/20 of 960, namely, 48. This is what goes into it in [the number of hundreds of] quadruple-heqats, (*4-ḥḳ3t*), [i.e.,] in **grains, 4800 heqats.**

Problem 42 is a quite similar problem and needs no elaboration here. It suffices to say that in both problems the cylindrical volumes are determined by finding the area of the circular base of the cylinder as a square in the manner already described above in the paragraphs on the area of a circle and multiplying that area by the height of the cylinder. In these problems the author first finds the volume of the cylindrical granaries in cubic cubits and then converts them into khar (where the khar is a cubic unit 2/3 of that of the cubic cubit) by multiplying the cubic cubits by 1 1/2, the khar being subsequently converted into heqat of grain. But in the Kahun fragment IV.3 (Document IV.3, Cols. 13-14) the volume of the cylinder in khar is immediately found without first finding it in cubic cubits:

[What is the volume in khar of a cylindrical granary whose diameter is 12 cubits and whose height is 8 cubits? As is evident from the calculations below, the procedure is to add 1/3 of the diameter to the diameter, multiply the total by itself; then multiply that result by 2/3 of the height, i.e., 5 1/3, to produce 1365 1/3 khar. I put the operations of Col. 14 first.]

[Col. 14]

[\ 1	12]
2/3	8
\ 1/3	4
Total	16.
\ 1	16
\ 10	160
\ 5	80
Total	256.

[Col. 13]

\ 1	256
2	512
\ 4	1024
\ 1/3	85 1/3
Total	1365 1/3 [khar].

The equality of the two different ways of finding the volume of the cylinder in khar can be demonstrated by expressing the differing procedures algebraically. In the method of Problems 41 and 42 of Document IV.1 the volume in khar is determined by the formula $V = (3/2)[h(d - d/9)^2]$ and in the Kahun problem the pertinent formula would be $V = (2/3)\, h\, (d + d/3)^2$. Both formulas reduce to $V = (32/27)\, hd^2$. Note further that in the Kahun problem, with its skeleton text of numbers, the volume in khar is not then converted into heqat of grain as in the problems found in Document IV.1. Incidentally, I shall not discuss here the corruption of the text of Problem 43 presented as the first version of that problem in Document IV.1 (with its difficulties mentioned there). But, like the Kahun problem, it also seems to have had as its objective the determination of the volume of a cylinder in khar directly, despite its superfluous mention of taking 1/9 of the diameter from the diameter. In fact, we should say here that, by following the procedure found in the Kahun problem, the revision of Problem 43 suggested by Gillings (the third version of that problem given in the translation of Problem 43 below) seems to preserve considerably more of the original text given in the Rhind Papyrus than does the revised second version presented below in Document IV.1.

The most impressive of the volumetric problems given in the Egyptian mathematical papyri is Problem 14 in Document IV.2, where the procedure for determining the volume of a truncated square pyramid, i.e., the volume of frustum of a pyramid with a square base, is given. I insert it here from Document IV.2 below:

[Col. XXVII]
[Lin. 1] **Example of calcnlating a truncated [sqnare] pyramid.**
[Lin. 2] If someone says to you: "A pyramid of 6 for the height (*šṭwtí*)
[Lin. 3] by 4 on the base (i.e., the side of the lower square) by 2 on the top (i.e., the side of the upper square)."
[Lin. 4] You are to square this 4; the result is 16.
[Lin. 5] You are to double 4 (i.e., multiply 4 by 2); the result is 8.
[Lin. 6] You are to square this 2; the result is 4.

[Col. XXVIII]
[Lin. 1] You are to add the 16
[Lin. 2] and the 8 and the 4;
[Lin. 3] the result is 28. You are to take
[Lin. 4] 1/3 of 6; the result is 2. You are to take 28 two times; the result is 56.
[Lin. 5] Behold, [the volume] is 56. You will find [that this is] correct.

[Col. XXIX]
[For the diagram given in this column with translated numerals and their computation, see Fig. IV.10.]

It is clear that the author has here used the correct formulation for the volume of a truncated square pyramid (i.e., the arithmetical steps specified by the problem's text are precisely those one would take when using the correct formula). That they should have been able to discover such a formula is not surprising in view of their great activity in building pyramids with the consequent need to know the quantity of blocks required for their construction. But how the ancient Egyptians derived the given procedure is not known. Still the method of discovering the formula has been much speculated on by modern historians of mathematics, as Peet and Gillings indicate[69] and my quotations here in the text and in the notes illustrate. Let us look at an ingenious solution proposed by Gunn and Peet:[70]

> By what means could the early Egyptians determine the volume of a truncated pyramid?
>
> How did the determination, when accomplished, furnish the method of calculation which we find employed?
>
> Now with one reserve, dealt with below, it is not a difficult matter to ascertain the volume of a given truncated pyramid by experimental means, namely by the fairly obvious method of division into parts and recombination of these parts into simple solids, the sum of whose volumes will give the volume of the frustum. Make a frustum of manageable size and of some easily cut substance, and divide it by four vertical cuts, each one coinciding, as to part of its length, with one side of the upper surface, as shown in Fig. 2 [*Author:* = my Fig. IV.9A, α], in which the frustum is seen from the top, the thin lines representing the downward cuts. The frustum will now have been cut into nine parts, numbered 1 to 9 in...[Fig. IV.9A, α]; these parts, all of which have the height of the frustum, fall into three classes:
>
> (a) Part No. 1, the central portion, a rectangular solid having the base equal to the upper surface of the frustum (...[Fig. IV.9A, β]).

(b) The four equal parts Nos. 2, 3, 4, 5, each a wedge, in section a right-angled triangle, and having a base one side of which is equal to a side of the upper surface of the frustum, and the other side of which is equal to half the difference between the sides of the lower and upper surfaces of the frustum (...[Fig. 9A, γ]).

(c) The four equal parts Nos. 6, 7, 8, 9. Each has two vertical surfaces and two oblique ones; it terminates at the top in a point, and its base, always square whatever be the proportions of the frustum, has the side equal to half the difference between the sides of the lower and upper surfaces of the frustum (...[Fig. 9A, δ]).

The combinations necessary to group these solids into larger and more regular ones become obvious after a few moments' manipulation. Taking any two of the wedges Nos. 2, 3, 4, 5, and turning one of them upside down, we find that they fit together into a rectangular solid having the same height and base as one of the wedges (...[Fig. 9A, ε]). The other two wedges being similarly fitted together, and joined to the first pair with all the wedges in single file, we obtain a rectangular solid having a base double that of one of the wedges and equal in height to these (...Fig. IV.9, ζ). To this we may now join part No. 1 [i.e., β], in single file with the wedges, for, whatever the proportions of the frustum, any vertical face of the square-based part 1 will have the same breadth as that of the vertical rectangular faces of the wedges, and the heights of all the parts are equal. We have now built up a rectangular solid equal in height to the frustum, and standing on a base the sides of which are respectively equal to the sides of the lower and upper surfaces of the frustum (...[Fig. 9A, η]).

Turning now to the four parts Nos. 6, 7, 8, 9, we find that if they are pushed together in the directions shown by the arrows in...[Fig. IV.9A, α] until all their vertical faces are hidden and in contact with each other, they constitute a true pyramid having the height of the frustum, and a

> square base the side of which is equal to the difference between the sides of the lower and upper surfaces of the frustum (...[Fig. IV.9A, θ])
>
> By a method remote from those of pure geometry, let us say by cutting up a lump of half-dry Nile mud with a piece of stout thread, we have now converted a frustum into a rectangular solid [i.e., a rectangular parallelepiped] and a pyramid. To find the volume of the former is of course an elementary matter, and provided that we also know how to determine the volume of the pyramid (the reserve made above [at the beginning]), we have only to add the two volumes to get that of the frustum.

Another somewhat similar but more crisply presented line of argument was given sometime later by B.L. van der Waerden, but also with the same reserve, namely that the volume of a pyramid was already known:[71]

> An outstanding accomplishment of the Egyptian mathematics is found however in the entirely correct calculation of the volume of the frustum of a pyramid with square base, as found in the Moscow Papyrus (Plate 5a [*Author* = my Fig. IV.6k, Cols. XXVIII-XXIX]) by means of the formula
>
> (1) $V = (a^2 + ab + b^2) \bullet h/3$,
>
> where h is the height and a and b the sides of the lower and upper base.
>
> It is not to be supposed that such a formula can be found empirically. It must have been obtained on the basis of a theoretical argument; how? By dividing the frustum into 4 parts, viz., a rectangular parallelepiped, two [right] prisms and a pyramid (see Fig. 5 [*Author* = my Fig. IV.9B, a]), one finds, the volume of a pyramid being assumed as known, the formula
>
> (2) $V = b^2h + b(a-b)h + (a-b)^2 \bullet h/3$.
>
> Neugebauer suspected that (1) came from (2) by means of an algebraic transformation.[72] But can one justify the as-

sumption that the Egyptians were able to make such an algebraic transformation? They were able to calculate with concrete numbers, but not with general quantities. This leads us to wonder whether in this case Egyptian arithmetic was influenced by Babylonian algebra. Or should we suppose that (1) was obtained from (2) by a geometric argument? One might imagine the following deduction.

For convenience, let us assume that one of the edges is perpendicular to the base. The two prisms of Fig. 5 (*Author:* = my Fig. IV.9B, a) are changed to rectangular blocks of half the height; the pyramid is also transformed into such a block, but having 1/3 of its original height (Fig. 6 [*Author* = my Fig. IV.9B, b]). Then the upper third of the first of these blocks is removed and placed on top of the second one (Fig. 7 [= my Fig. IV.9B, c]). The solid that is obtained in this way can be divided into 3 horizontal layers, each of which has the height $h/3$; the lower one of these layers has a base equal to a^2, the middle one has a base ab and the upper one a base b^2.

This derivation of the formula does not transcend the level of Egyptian mathematics. But, I certainly do not want to tell a fairy tale, and I definitely do not assert that the Egyptians actually proceeded in this manner. There are indeed other possibilities. For example, Cassina [*Periodico di Matematica* (4a seria) Vol. 22, pp. 1-29] has suggested another derivation of the formula for the special case [appearing in the Moscow Papyrus] (and this indeed is the only case dealt with in the papyrus) in which the area (! *should be* side) of the upper base is one half of that of the lower base. Moreover, we should not a priori eliminate a possible effect of Babylonian algebra.

Whichever one of these hypotheses is adopted, we must suppose that the Egyptians knew how to determine the volume of a pyramid. [*Author:* Note that I have corrected the common misspellings "frustrum" and "parallelopiped" used in the long passage just quoted.]

One further point worth mentioning in this argument is that van der Waerden has for the sake of convenience started with a frustum one of the edges of which is perpendicular to the base (and although one could perhaps interpret the figure given in the hieratic text of Problem No. 14 as being the frustum of such a pyramid, the general understanding of the relevant pyramid here and elsewhere is of the ordinary pyramid found constructed in Egypt, and is certainly so of the pyramids whose slopes are found in Problems 56-59 of Document IV.1 discussed below). At any rate, using a frustum of a pyramid with a perpendicular edge allows van der Waerden to simplify his argument by dealing with two wedges (not four), one central parallelepiped, and one pyramid. Also both these arguments (those of Gunn-Peet and van der Waerden) seem to grant that the general formula appearing in Problem 14 (van der Waerden's formula No. 1) was not derived by means of *algebraic* transformation from formula No. 2, as Neugebauer suspected. Rather they assume that some kind of geometrical, that is graphic, transformation was employed. But the crucial difficulty in all of such graphical solutions is that they depend on prior knowledge by the Egyptian mathematician of the formula for the volume of a pyramid, namely $V = 1/3$ times the product of the base and the altitude. Gunn and Peet faced that problem, though their graphic solution of it is not too convincing:[73]

> Now there is no direct evidence, from the mathematical documents or other sources, that the Egyptians knew the very simple calculation required to determine the volume of a pyramid; yet it is almost inconceivable that they did not. Being accustomed, from the Third Dynasty onwards, to construct large pyramids in stone and brick, and it being of the greatest importance to know in advance the amount of material, and hence of labour and time, that these buildings would require, they will certainly have made every effort in their power to solve the problem. Here again experiment yields the secret. If, again with Nile mud and a thread, we attempt to find it by dividing a model pyramid

into parts and combining these, no useful results follow, because there will always be polyhedra which refuse to make up into simpler solids. But the obvious way to make a small pyramid of some fairly soft substance is to take a rectangular solid on a square base (parallelepiped) and with two slanting downward cuts passing through the middle line *OO′* of its upper surface to separate it into a central triangular prism with a wedge on each side of it, as shown in ...[Fig. IV.9C, a]. Next, without removing the wedges from the prism, make two similar cuts passing through the *XX′* at right angles to *OO′*. The result will be to divide the whole solid into nine parts. In the centre will remain a pyramid, visible in...[Fig. IV.9C, b], on a square base, and of the same height as the original parallelepiped. Resting against its four sides we shall have four equal tetrahedra, two of which are shown detached in...[Fig. IV.9C, b]. Between each pair of these tetrahedra is another tetrahedron (e.g., *OPX′B′B* in...[Fig. IV.9C, a]) whose vertex *B* is one of the corner points of the square base of the original figure, whose base is a square *OPX′B′*, forming a quarter of the upper surface of the parallelepiped, and one of whose edges *BB′* is at right angles to this base. The position and shape of these last four tetrahedra can easily be imagined from....[Fig. IV.9C, c], where they have been removed, leaving only the central pyramid with the other four tetrahedra attached to its sides.

Now it is obvious that those four corner tetrahedra, *OPX′B′B*, etc., can be fitted together (exactly in the manner of those which we obtained by cutting up a frustum of a pyramid, p. 180 [and quoted above over note 68]) to form a pyramid precisely similar and equal to that left in the centre of the parallelepiped. We have thus dissected our parallelepiped into two equal and similar pyramids and four tetrahedra. This result may well have suggested to the Egyptian the possibility of some constant relation between the volumes of the pyramids and that of the parallelepiped from

> which they were cut, and since the four remaining tetrahedra cannot be combined into any simpler solid we may suppose that he had recourse to weighing, which would at once reveal the fact that the four tetrahedra are together equal to each of the two pyramids. Consequently each pyramid is one-third of the original parallelepiped in volume, or V = (height x base) / 3.

The authors then go on to recount a further simple experimental way suggested by R. Engelbach that the formula for the frustum of the pyramid given in Problem 14 might have been discovered, which I suggest that the reader might pursue on his own.[74] I also recommend that the reader read the critique of the Gunn and Peet article made by K. Vogel in the same journal.[75]

We mentioned above that the Egyptians were interested in problems concerning pyramids because of their great activity in building them. Among such problems were those concerning their slopes. They would have had to know the desired slope of a pyramid before building it. Problems 56-59 found in Document IV.1 are concerned with the slopes of pyramids as determined by their vertical height and the length of a side of the square base.

In illustration of this determination of the slope or *sḳd* (always transcribed as "seqed" rather than "seked" in my volume), let us quote the first of these problems, Problem 56 from Document IV.1 below:

Example of reckoning a pyramid (*mr*) whose base-side (*wḫꜣ-ṯbt*) is 360 [cubits] and whose altitude (*pr-m-wś*) is 250 [cubits]. Cause that I know (i.e., calculate) its seqed (i.e., slope). [See Fig. IV.2mm, Plate 78.]

Take 1/2 of 360 and the result is 180. Multiply 250 so as to find 180. It makes 1/2 1/5 1/50 of a cubit. A cubit is 7 palms. Multiply 7 as follows:

1	7
1/2	3 1/2
1/5	1 1/3 1/15
1/50	1/10 1/25.

The seqed is 5 1/25 palms.

If we use the figure of pyramid on a square base redrawn by Peet [*Author:* see my Fig IV.9F], the slope or seqed is, of course, determined by the ratio of one half of a side of the base to the altitude of the pyramid, namely, the slope of angle *GFD*. The slope is what we would call today the cotangent of that angle, i.e., the ratio *GF/GD* in Peet's figure. Of course, the slope was not only crucial for the construction of pyramids, as noted here, but also for the construction of water clocks shaped as sections of inverted cones (see Volume 2 of my work, pp. 65-77 and *passim*). We should note in this connection that Problem 60 of Document IV.1, which is said to be the determination of the seqed of a pillar or column *(iwn)* may well be in fact the finding of the slope of a cone (see Document IV.1, notes 88-90):

[In] **a pillar** (*iwn*) [*or perhaps* a cone?] with a base-side (*sntt*) [*or perhaps* a diameter?] of 15 cubits and a height of 30 [cubits], what is its seqed? [See Fig. IV.2oo, Plate 82.]
Take 1/2 of 15; it is 7 1/2. Operate on 30 so as to get 7 1/2. The result is 1/4, which is the seqed.

One piece of evidence that seems to favor the interpretation of this problem as referring to the slope of a cone is the fact that the diagram accompanying the hieratic text (Fig. IV.2oo, Plate 82) appears to be that of a cone. However, *iwn* ordinarily is a column or pillar.

There is one final problem that appears to be in the realm of solid geometry, being concerned with the surface of a "basket." It is Problem 10 of Document IV.2 and has caused considerable discussion, for the editor of the Moscow Papyrus, W.W. Struve, believed it to be the correct solution for the area of a hemisphere. Let me first quote the text of that problem as reconstructed by Struve:

[Col. XVIII] *[cont.]*

[Lin. 1] **Example of calculating a basket** (⌓ |, *nbt*) [assumed by Struve as hemispheric in shape; see Fig. IV.8]
[Lin. 2] If someone says to you: "A basket with a mouth opening
[Lin. 3] of 4 1/2 (i.e., a diameter of this size) in good condition (*ʿḏ*), oh
[Lin. 4] let me know its [surface] area (*ꜣḥt*)."
[Lin. 5] [First] calculate 1/9 of 9, since the basket is
[Lin. 6] 1/2 of an egg-shell (*? ỉnr?*). The result is 1.

[Col. XIX]
[Lin. 1] Calculate the remainder as 8.
[Lin. 2] Calculate 1/9 of 8.
[Lin. 3] The result is 2/3 1/6 1/18. Cal-
[Lin. 4] culate the remainder from these 8 after
[Lin. 5] taking away those 2/3 1/6 1/18. The result is 7 1/9.

[Col. XX]
[Lin. 1] Reckon with 7 1/9 four and one-half times.
[Lin. 2] The result is 32. Behold, this is its area.
[Lin. 3] You will find that it is correct.

For the most part the judgment of historians of mathematics has gone against Struve's interpretation, all of the critics being so conscious of the brilliance of the later Greek discoveries in the realm of spherical geometry that they were apparently reluctant to accept any significant role in those discoveries by Egyptian calculators. Peet's critical philological rejection of Struve's text, which I have given below in note 18 of Document IV.2, seems to have reinforced the opinion of the doubters. On the other hand, not all of Peet's textual criticisms seem to me to be entirely just, particularly those connected with Peet's convenient additions to the text that are not all supported in the hieratic text, as I said in the same footnote cited above. While Peet believed he had overturned Struve's interpretation, his suggested reinterpretations are far from sure, as I further remark in note 19 of Document IV.2. Of his own two differing interpretations, only the one which interprets the problem as finding the surface area of a half cylinder (i.e., a cylinder split vertically) has some plausibility, and thus I have included it as a second version of Problem 10 in the text of Document IV.2 below. But as we show toward the end of note 18 of Document IV.2, R.J. Gillings, while not being dogmatic, is so struck by what he thinks is the conformity of the Egyptian procedure with the correct formula for the surface of a hemisphere, that he is inclined to accept Struve's conclusions, a position I sympathize with. However, he makes no effort to answer Peet's philological criticism of Struve's interpretation. For a comparison of the differing interpretations of Struve and Peet, see my Fig. IV.38 taken from Neugebauer's *Vorlesungen über Geschichte der Antiken mathematischen Wissenschaften*,[76] where their respective corrections of the text are given within square brackets. Neugebauer also suggests there (see note 76) that the problem may pertain to the area of a dome-shaped granary.

Conclusion

I have attempted to illustrate and analyze the main features and procedures of ancient Egyptian mathematics in the foregoing pages. The reader will no doubt have been impressed by the prevailing dominance of calculation everywhere in the ancient papyri, calculation that was often brilliantly executed within the limits of their arithmetical and notational conventions. The dominance of calculation is true of the problems whether they refer to the simple arithmetical procedures or whether they are concerned with plane or solid geometry. Such arithmetical practices, whether or not they are presented as specific examples of what were common-place and apparently general procedures (which they were for the most part in the mathematical papyri), show the intimate connection throughout between the problems presented and the needs and techniques of measurement. Lurking behind the calculations that exemplify the apparently general procedures, even if they are invisible, may have been graphic procedures, perhaps like those suggested above in the sections on geometry. These graphic procedures perhaps performed the same function as the algebraic transformations that some modern historians of mathematics thought they detected in the Egyptian solutions of geometric problems. Once achieved in this realm of mathematics, similar arithmetical procedures involving manipulation of the equations found in Aha problems and in many other of the numerical problems may well have been adopted and even become common place.

The use of graphic techniques to find the general procedures so often implied in the mathematical problems we have discussed in some detail in this chapter, despite Guillemot's doubts, seems appropriate, one may almost say analogically confirmed, by the manifold use of scaled grids (a geometrical graphic tool itself) to transform the standard measurements of body parts into a regular canon of proportions for the creation of the artistic representations, the paintings and sculptures of human figures, found everywhere in the tombs and temples of Ancient Egypt.

In the next and concluding volume of my work, I shall attempt to examine this canon and its formation for the light it throws on a general investigation of Egyptian modes of viewing and depicting nature.[77]

I have been rather cautious about problems that imply generality, using such adjectives as "invisible" or "implied." Nevertheless, I believe firmly that generality is what the mathematical papyri are all about. There simply is no doubt that the authors intended for the readers who wished to compute problems of the same nature as those given in the papyri to turn to these problems in the papyri even though their data differed. While general rules are ordinarily not written out, we must remember that the so-called Problem 61B of Document IV.1 was in fact a general rule for taking 2/3 of an unitary odd fraction, i.e., of the reciprocal of an odd number: take the reciprocals of the products of 2 times the odd number and of 6 times the odd number, and these two unit fractions together give the desired solution. Also the reader should note the author's conclusion to Problem 66 of Document IV.1: "You shall proceed in this way [given above] in any example like this." Consideration of the generality of Egyptian mathematics has led to rather inconclusive discussions of how close to modern "science" (by which the historians seem to mean modern "mathematics") the Egyptians came.[78] I have not always escaped such a temptation in these pages, but my principal purpose in this chapter, and indeed in the documents below, has been to delineate such similarities and divergencies as existed. Most of all, I have attempted to show in sufficient detail the actual mathematical procedures followed by these early mathematicians as they went about measuring and computing to help satisfy their manifold needs.

Notes to Chapter Four

[1] For mention of the Egyptian "rope-stretchers" or surveyors, see Democritus' claim that no one surpasses him "in the construction of lines with proofs, not even the rope-stretchers *(harpedonaptai)* among the Egyptians." See B.L. van der Waerden, *Science Awakening* (Groningen, 1961), p. 15. As I note in the text below, the surveyors are often mentioned as laying out temples in the *Early Annals*. One of the earliest embracive treatments of length measurement in Egypt and elsewhere in the ancient world was that composed by C.R. Lepsius, *Die Längenmasse der Alten* (Berlin, 1884). This book was an outcome of Lepsius' earlier study of the Egyptian cubit published in 1866 (see note 4 below). His *Die Längenmasse* is still of interest to the student of ancient measurement, and it will surely repay the reader who consults it. See the recent succinct account on the the metrological units found in third millennium B.C. documents by James Ritter, "Metrology and the Prehistory of Fractions," *Histoire de fractions, fractions d'histoire*, coordonné par P. Benoit et al. (Basel, Boston, and Berlin, 1992), Chapt. 1, pp. 25-34.

[2] Herodotus, *The Histories*, translated by Aubrey de Sélincourt (Penguin Books, 1954), Book II, Sect. 109, pp. 141-42. The king to whom Herodotus attributed this act was a Sesostris, who was, however, misplaced before the pyramid builders of the fourth dynasty. Indeed his account of the deeds of this Sesostris was a mélange of the activities of the various kings that belonged to the twelfth dynasty.

[3] T.E. Peet, *The Rhind Mathematical Papyrus* (London, 1923), pp. 9-10.

[4] The section on measures has been based on the documents that accompany this chapter and several important discussions: C.R. Lepsius, *Die altägyptische Elle und ihre Eintheilung* in *Philologische und historische Abhandlungen der königlichen Akademie der Wissenschaften zu Berlin. Aus dem Jahre 1865* (Berlin, 1866); pp. 1-63, 64*-64****, and 5 Tafeln; F. Ll. Griffith, "Notes on Egyptian Weights and Measures," *PBSA*, Vol. 14 (1892), pp. 403-50, Vol. 15 (1893), pp. 301-16; Peet, *op. cit.* in the preceding note, pp. 24-26; A.B. Chace, et al., *The Rhind Mathematical Papyrus*, Vol. 1 (Oberlin, Ohio, 1927), pp. 31-34; Gardiner, *Egyptian Grammar*, pp. 197-200; and the exceedingly useful articles of W. Helck and S. Vleming on "Masse und Gewichte," *Lexikon der*

Ägyptologie, Vol. 3 (Wiesbaden, 1980), cc. 1199-1214, the article of Helck's covering measures in Pharaonic times (cc. 1199-1209) and that of Vleming's Demotic texts in Ptolemaic Times (cc. 1209-1214). Copious citations are given in the last two articles.

[5] For other estimates, see the measurement 52.5 cm. quoted in the article by Helck given in note 4, Vol. 3, c. 1200, and the discussion in the work of Gillings, *Mathematics in the Time of the Pharaohs*, p. 207, n.*. As Griffith observes in *op. cit.* in note 4 above, p. 406: "According to Mr. Petrie, the cubit shown in the marvelously accurate work of the Great Pyramid is 20.62 inches, and the average of the royal cubit on the rods is 20.65." See the detailed discussion of linear measures in W.M.F. Petrie, *Ancient Weights and Measures* (Warminster, Wiltshire, England, 1926), pp. 38-41. Also see note 8 below.

[6] As reworked in English by Griffith, *op. cit.* in note 4, p. 404, from Lepsius, *op. cit.* in note 4, pp. 43-44.

[7] The description quoted here is from the Museum of Fine Arts, *Egypt's Golden Age: Catalogue of the Exhibition* (Boston, 1982), the catalogue of an Exhibition at the museum, Feb. 3-May 2, 1982, photograph no. 30, with accompanying description, bibliography, and literature.

[8] See Petrie, *Wisdom of the Egyptians* (London, 1940), p. 71: "The half diagonal of this [royal cubit of 20.6 ins.] was the *remen*, a second unit of 14.6 ins., which was divided in 20 digits of .73 [ins.]. Thus, by the use of the diagonal, the half of any square area could be readily formed and defined. That this was fully recognized is shown by the half of the area of 100 x 100 cubits being also called *remen* in land measure.... The *remen* means an arm, or branch of a tree, and agrees with the fore-arm down to the clenched knuckles, still a favourite mode of measuring in Egypt." In the same work, p. 10, he speaks of the 20.62 cubit (also see n. 5).

[9] Griffith, *op. cit.* in note 4 above, p. 404. See also my next note.

[10] W.C. Hayes, *The Scepter of Egypt*, Part II (Greenwich, Conn., 1959), pp. 412-13. These fragments mentioned by Hayes and other cubit-rods were the object of some interesting remarks by G. Sarton, "On a curious subdivision of the Egyptian cubit," *Isis*, Vol. 25 (1936), pp. 399-402. He says (pp. 401-02) concerning the division of the digit into 16 parts in the examples of cubit-rods given by Lepsius and Schiaparelli: "It should be noted that as the length of a digit is less than 2 cm., subdivision into sixteen parts reaches almost the practical limit of visibility.

"What may have been the purpose of that strange subdivision? Why was it necessary to have ready scales in fractions of a digit–from the half to the sixteenth? This was probably connected with the Egyptian exclusive interest in fractions of the type 1/n. [see the Table of Two in my Document IV.1 below]. There rulers made it possible to determine the actual length indicated by such an expression as one cubit plus one fifth, one eighth and one fourteenth, yet the same purpose might have been attained in a simpler manner.

"The use of such rulers for the graphical solutions of arithmetical problems cannot be countenanced, for the divisions were not precise enough. Indeed as Lepsius remarked (1866, 18) the subdivisions of the cubit are sometimes unequal and the digits have not all the same length. In the schematic example on his plate I [*Author* = my Fig. IV.24 (Tafel 1)], the first sixteen digits are 18.75 mm. long, the eight following (17th to 24th), 17.9 mm. long, the last four, 21.87 mm. long. Indeed various characteristics of the cubits preserved in our museums suggest that they were objects meant for ceremonial rather than practical use. For example, in the beautiful [golden] cubit of the museum of Torino [see below], each digit is associated with a god whose name is written above it. Moreover various cubits being made of stone were too heavy and fragile for convenience."

Let me quote at some length from E. Schiaparelli's description of the golden cubit rod of Cha [or Kha], *La tomba intatta dell'architetto Cha nella necropoli di Tebe* (Torino, 1927), pp. 168-172: "Fra gli oggetti rinvenuti nella tomba di *Cha* ve ne sono alcuni che, a giudicare dalle iscrizioni incise o scritte sui medesimi, si deve credere che sieno stati a lui donati o dal Faraone o da persone amiche. Tali sono un cubito di lamina d'oro, due vasi di bronzo....

"Il cubito di lamina d'oro è oggetto di grandissimo pregio (*Figg. 155* e *156* [=Figs. IV.27b and IV.27c below]). Noi lo trovammo fasciato con una benda di fine e soffice tela in fondo al cofano nel quale erano stati riposti i sette vasi di alabastro pieni di olio. Levati questi, sul fondo del confano, in mezzo ad abbondanti cenci di tela, vedemmo luccicare qualcosa, ed era questo bellissimo prodotto della oreficeria egiziana. È, come si disse, di lamina d'oro, sostenuta internamente da anima di legno: si apre alle due estremità con due testate, che si possono levare e mettere a piacimento, e sulle quali è impresso in rilievo il cartello reale di Amenofi II [Amenhotep II], sostenuto da un *Rechit* o uccello con braccia umane (*Fig. 154* [=Fig. IV.27d below]). Intorno alle testate corre un fregio a cordone finemente lavorato a sbalzo, e tutta intera la superficie del cubito è coperta di iscrizioni leggermente rilevate e cesellate, ovvero dei segni indicanti le varie suddivisioni del cubito." Then follows a description of the Royal cubit and the small cubit and their divisions much like the table I have given from Lepsius in the text above and where they are given on the rod. Now I continue with Schiaparelli's account of the cubit (p. 169):

"La faccia superiore del nostro cubito, nonchè quella inferiore e quella di dietro sono coperte con tre distinte e separate iscrizioni; la seconda delle quali celebra le vittorie di Amenofi II contro il *Naharina* (la Mesopotamia) e contro i Negri (*Fig. 156* [=Fig. IV.27c below]), la prima accenna alla gioia del popolo egiziano pel valore del Faraone (*Fig. 155* [=Fig. IV.27b]), e l'ultima è particolarmente interessante per le indicazioni specifiche che contiene." After translating and discussing the historical information in the inscriptions, he reasons (p. 172) that this cubit was prepared for the Pharaoh and then was given to Cha.

"Oltrechè per la ricchezza della materia e per la finezza del lavoro, principalmente per le iscrizioni che vi sono impresse, questo cubito non può adunque considerarsi come un oggetto di carattere e di uso privato, ma è evidentemente un oggetto commemorativo, un vero monumento, di pertinenza del Faraone e preparato per esso: e se il medesimo si sia trovato nella tomba di *Cha*, è a ritenere che a lui sia stato donato da Amenofi II stesso."

Schiaparelli (p. 80) also refers to a quite different, practical working cubit-rod of acacia wood which was divided into two pieces hinged together for easy portability, with a picture of it (Fig. 47 of that work = Fig. IV.27e below). Presumably its owner, as an architect, would have made much use of it. Both this and the golden cubit-rod are on exhibit in the Museo Egizio at Torino.

[11] Hayes here quotes the earlier description by N.E. Scott, "Egyptian Cubit Rods," *Bulletin of the Metropolitan Museum of Art*, n.s., 1 (1942), p. 72 (full article, pp. 70-75).

[12] Scott, *ibid.*, p. 73.

[13] *Die altägyptische Zeitmessung* (Berlin and Leipzig, 1920), pp. 14 and 27-28, where Borchardt gives tentative suggestions, first with respect to the theory of the use of ceremonial cubit-rods in connection with outflow water clocks (p. 14) and then with respect to their use in connection with shadow clocks (pp. 27-28). When I described outflow water clocks and shadow clocks in Volume Two, I did not discuss these cubit-rods and how they might have been used for quickly determining variable hour lengths in different months by noting volumetric variations of the outflow water or the differences of shadow lengths on shadow clocks at different times of the year because of the thinness of the evidence for Borchardt's speculations. But, the argument runs, variation of volumetric quantities of the outflowing water in the case of the tables which perhaps pertain to water clocks was assumed because the measures expressed were in terms of the repeated use of the three sums of some Horus-eye fractions (i.e., 1/4 + 1/8 + 1/64, 1/4 + 1/8, and 1/4 + 1/32 + 1/64; for Horus fractions see below, Fig. IV.3, and Document IV.1, n. 46), and these fractions were ordinarily applied only to fractions of a heqat, i.e., a measure of the content of a vessel, usually of grain but perhaps also of a liquid like water. Furthermore, there is a distinct reference to "The hour (*wnwt*, in Fig. IV.27a, cubit-rod no. 3, first word of register 1) according to the cubit-rod," followed by the apparent mention of a jar (?) "filled with water." In the case of the tables suggested as applying to shadow clocks, the measures were all in terms of cubits and palms, i.e., linear measures, as they would be in comparing shadow lengths at different months of the year (e.g., see Fig. IV.27a again, but this time to cubit-rod no. 2, just before the middle of register 1).

[14] Helck, *op. cit.* in note 4 above, cc. 1200-1201. In addition to those quoted here, Helck gives other less frequently used measures. Note that Griffith, *op. cit.* in note 4, pp. 410-20, and the "Table of Multiples and Subdivisions of the *Set* or *Arura*" gives the value of the *ḫȝ* (found, I have said, in the *Annals on Stone*) as 10 arouras, i.e., 1000 cubit areas, rather than as 10 *tȝ, i.e.* 1000 square cubits, the measure given by Helck. See also Gardiner, *op. cit.* in note 4, p. 200, who gives examples of the *ḫȝ* as a 10-aroura measure. He also remarks that it is more fully written as *ḫȝ-tȝ*, i.e., 1000-cubit-strips or "1000-cubit-areas." He estimates the *sṯȝt* as "2735 square metres or roughly 2/3 acre." I have given the fractions of the aroura found in the *Annals on Stone* on page 4 of this volume and in Fig. I.50 of Volume One of my work. See also Gardiner, *ibid.*: *rmn* = 1/2 *sṯȝt*, *ḥsb* = 1/4 *sṯȝt*, *sȝ* = 1/8 *sṯȝt*, and a cubic strip or *tȝ* = 1/100 *sṯȝt*.

[15] A.B. Chace, editor, *The Rhind Mathematical Papyrus,* Vol. 1 (Oberlin, Ohio, 1927), pp. 33-34.

[16] I take this valne of the *hin* in liters from Helck, *op. cit.* in note 4, c. 1201. Gardiner, *Egyptian Grammar,* p. 199 (Sect. 266), notes that actual inscribed examples average about .503 liters. Notice that Helck gives the value of the *khar* as 10 *heqat,* while in Problem 41 it is given as 20 *heqat.*

[17] Chace, *op. cit.,* Vol. 1, pp. 31-32. On pp. 32-33, he remarks that the author of the Rhind Papyrus asserts the *khar* to be 2/3 the cubic-cubit, and says that 20 *khar* "make 100 quadruple *hekat* or 400 simple *hekat.* This would make the *hekat* 292.24 cubic inches, as stated above." For further details and bibliography on the corn-measure and other measures of content, see Gardiner, *op. cit.* in note 4, pp. 197-99.

[18] I call to the reader's attention the detailed list of measures given by Helck in the article cited in note 4, cc. 1203-05. The exact valne of many of these in relationship to the standard measures cannot be determined.

[19] See K. Sethe, *Von Zahlen und Zahlworten bei den alten Ägyptern und was für andere Völker und Sprachen daraus zu lernen ist* (Strassburg, 1916), pp. 60-108, and particularly, pp. 83-89.

[20] Gardiner, *op. cit.* in note 4 above, p. 196. In a short communication on the supposed unacceptable practice of repeating a unit fraction in a series of unit fractions mentioned by Gardiner in this passage, David Silverman notes a possible exception in *JEA,* Vol. 61 (1975), pp. 248-49. See also my comments in the passage following the Gardiner quotation, which comments suggest a cogent mathematical reason for discounting Gardiner's view that the practice of not repeating a unit fraction was a general one. It is true that the rule or rather practice is followed in the Table of Two, but obviously not everywhere, as is evident in the writing of the equalities in Document IV.5. Also see the interesting effort by M. Caveing, "Le statut arithmétique du quantième égyptien," *Histoire de froctions, fractions d'histoire,* coordonné par: P. Benoit et al.

(Basel, Boston, and Berlin, 1992), pp. 46-50, to explain how the Egyptians' implicit use use of "proportion" and their concept of fractions as divisions anticipated the later coherently expressed views of the Greeks.

[21] See Peet, *op. cit.* in note 3, pp. 17-18, for an evaluation of the Egyptian procedure given in Problem 32 of Document IV.1, a procedure that resembles the modern method of a common denominator to accomplish the addition of fractions, and his somewhat critical judgment of the views of F. Hultsch, "Die Elemente der ägyptischen Theilungsrechnung. Erste Abhandlung," *Abhandlungen der philologisch-historischen Classe der Königlich-Sächsischen Gesellschaft der Wissenschaften*, Vol. 17, no. 1 (Leipzig, 1895), p. 112, and L. Rodet, "Les prétendus problèmes d'algèbre du manuel du calculateur égyptien (Papyrus Rhind)," *Journal Asiatique*, Series 7, Vol. 18 (1881), pp. 196-215 (whole article, pp. 184-232, 290-459), which attempt to differentiate the Egyptian and modern techniques. For Peet the general principle of both techniques is the same. When presenting the essential parts of the summation in the proof of Problem 32, Peet (pp. 17-18) he says "Here the Egyptian employs a method which at first sight appears to be that of a common denominator . All the fractions or aliquot parts seem to be reduced to terms of the highest aliquot part, namely the 912th part....

"This method [of summing unit fractions] obviously differs in small details from the modern method of common denominator. For instance, we always choose as our denominator the smallest number into which all the separate denominators will divide integrally. The Egyptian, unfamiliar with the principle of factors, often used a denominator which was smaller than the L.C.M., and consequently had fractional quotients (rarely involving smaller fractions than 1/8). It is hardly necessary to mention the further difference that the numerators of the fractions to be added were all unity.

"These, however, are distinctions of mere detail, and despite them the general principle might be the same. This is denied by both Hultsch and Rodet." Peet then goes on to present succinctly the views of these authors as expressed in the works mentioned at the beginning of this note, concluding finally (p. 19): "The fact is that both Hultsch and Rodet have been deceived by notation. There is and can be only one way of adding fractions, though there may be several ways of writing down the process. The fractions 1/4 and 1/5 are quite irreconcilable as they stand, and we can only combine them by reducing them to some smaller part of unity of which they are both multiples. We may do this in the modern way by means of the common denominator 20, or we may do it in the Arab way by means of the *mokhrag* 20. Avoid the notation 5/20 as we [and the Egyptians] may, we cannot in the end escape the fact that the 20 really stands for the twentieth part of some unit, and that the 5 is 5-twentieths of that unit. The process as seen in the Rhind papyrus is particularly deceptive since all the complicated additions there used are in the nature of proofs, i.e. the result is known to be some very simple aliquot part, *e.g.* 1/4 or 1/8, and though our denominator or *mokhrag* may be 960, the fact that we are really working in 960ths is apt to be overlooked or forgotten when the addition comes to 240 or 120, and the 960 drops out of sight, leaving only a simple 1/4 or 1/8."

[22] See the Introduction to Document IV.1 where the early editions of the Rhind Papyrus have been mentioned. Here I simply call attention to the first edition by A. Eisenlohr, *Ein mathematisches Handbuch der alten Ägypter (Papyrus Rhind des British Museum) übersetzt und erklärt*, Vol. 1 (Leipzig, 1877; 2nd ed., 1891, though 1877 is still on the title page). Hultsch, *op. cit.* in note 21, pp. 110-45. See also the later discussions of the red auxiliaries and common denominators from the article of van der Waerden quoted below over note 33 and in Gillings, *op. cit.* in note 5, pp. 81-88 and 251-53. The manner in which red auxiliaries are used in various contexts such as in "completion" problems and in problems for finding unknowns is also very thoroughly described by E.M. Bruins, "On Some Hau-Problems," *Janus*, Vol. 70 (1983), pp. 229-62 *passim*. For an example of their usage in a computation found in an ostracon, see also note 35 below. See the more recent account of Egyptian unit fractions, which includes a discussion of the red auxiliaries presented by Maurice Caveing, *op. cit.* in note 20, pp. 39-48 (full paper, pp. 39-52).

[23] Griffith, "The Rhind Mathematical Papyrus," *PSBA*, Vol. 13 (1891), pp. 328-32; Vol. 14 (1891), pp. 26-31; Vol. 16 (1894), pp. 164-73, 201-08, 230-48, and particularly pp. 201-08 of Vol. 16, which includes the quotation given in the text above. One should also read the early and able treatment of the Table of Two by Hultsch, *op. cit.* in note 21, pp. 175-87, who along with Griffith was one of the premier students of Egyptian mathematics writing in the first generation after the publication of Eisenlohr's edition of the Rhind Papyrus.

[24] I have substituted *Stammbrüche* for Griffith's *stamm-brüche* in this quotation, since Hultsch, Neugebauer, and other German authors use the form I have here adopted.

[25] Peet, *op. cit.* in note 3, pp. 33-47.

[26] Chace, *op. cit.* in note 15, Vol. 1, pp. 16-22.

[27] The Egyptian word is *ḏ3t*, which means "remainder" or "balance" or "deficiency."

[28] Chace, *op. cit.*, p. 20.

[29] O. Neugebauer, "Zur ägyptischen Bruchrechnung," *ZÄS*, Vol. 64 (1929), pp. 44-48; Neugebauer, *Arithmetik und Rechentechnik der Ägypter*, *QSGMAP*, Abt. B: Studien, Vol. 1 (Berlin, 1931), pp. 348-80; Neugebauer, *Vorlesungen über Geschichte der antike mathematischen Wissenschaften*, Vol. 1: *Vorgriechische Mathematik* (Berlin, 1934), pp. 137-65, and Q. Vetter, "Egyptské Dèleni," *Société Royale des Sciences de Bohème*, Classe des Sciences, 1921-22, No. 14. See also E.M. Bruins, "Ancient Egyptian Arithmetic: 2/N," *Kon. Nederland Akademie Wetanschappen*, Ser. A, Vol. 55 (Amsterdam, 1952), pp. 81-91; K. Vogel, *Vorgriechische Mathematik*, Teil 1: *Vorgeschichte und*

Ägypten (Hannover, 1958), pp. 38-44; Vogel, *Grundlagen der ägyptischen Arithmetik* (Munich, 1929); Gillings, *op. cit.* in note 5, pp. 45-80; and B.L. van der Waerden's older paper, "Die Entstehungsgeschichte der ägyptischen Bruchrechnung," *QSGMAP*, Abt. B: Studien, Vol. 4 (1938), pp. 358-82.

[30] B.L. van der Waerden, "The (2:*n*) Table in the Rhind Papyrus," *Centaurus*, Vol. 24 (1980), pp. 259-74. This is a shorter version of the author's earlier paper mentioned in the preceding note. Note that van der Waerden expresses unit fractions in the manner proposed originally by Neugebauer, i.e., the reciprocal form with a bar or macron over the denominator as a replacement for the unit numerator, and two bars over 3 to replace the numerator 2 in 2/3. But I shall substitute the ordinary fractional forms 1/n and 2/3 so that I do not continually have to use the more cumbersome Equations program in MSWord.

[31] *Ibid.*, p. 264. Note that, in regard to the source of the fourth equality of the second sequence, I have given in Document IV.5 (Col. 3, line 17) the more obvions reading which leaves the left-hand members as the scribe of the second copy thought that they should be, but corrects the right-hand member to 1/14 from 1/13.

[32] van der Waerden, *ibid.*, pp. 265-67.

[33] *Ibid.*, pp. 272-74.

[34] Gillings, *op. cit.* in note 5, p. 49.

[35] While this precept is generally valid for the Table of Two, there are exceptions in other documents, like Ostracon 153 found below the Theban tomb (No. 71) of Senmut, the supervisor of Hatshepsut's grand temple. W. Hayes edits and discusses this ostracon briefly in his *Ostraca and Name Stones from the Tomb of Sen-Mut (No. 71) at Thebes* (New York, 1942), pp. 29-30, and Plate XXIX. The computation there includes the multiplication in successive lines of 1/7 by 1, 2, and 4. Note that the multiplication of 2 x 1/7, which is equivalent to the division of 2 by 7, yields the answer as the sum of the three unit fractions 1/6 1/14 1/21 instead of the two unit answer 1/4 1/28 given in the Table of Two. Gillings, *op. cit.* in note 5, p. 87, notes the differing solution and discusses the computation of it as an example of the use of red auxiliaries. I give here the computation, with the red auxiliaries in bold face as usual (see Fig. IV.32 for a photocopy of the ostracon and its hieroglyphic transcription below):
:

1	1/7		
	3		
2	1/6	1/14	1/21
	3 1/2	**1 1/2**	**1**
4	1/2	1/14	
	10 1/2	**1 1/2.**	

Gillings explains this by suggesting the steps that the scribe followed (once more I change Gillings' notation of unit fractions by a macron over the denominator into unit fractions with numerator 1 and a slant bar before the denominator): "Whoever inscribed the ostracon was doing just what the scribe of the RPM [=Document IV.1] did in problems 28, 32, 36 and several others. The red 3 beneath the 1/7 means 'Take 3 as a multiplier of 7, to give the reference number 21.' He then multiplied the 2 (of line 3) by his multiplier [3] to give 6 which he then partitioned as 3 1/2, 1 1/2, and 1, each of which divides the reference number 21 in integers, and wrote them in red (line 4). These are the red auxiliaries.

"The scribe of the ostracon then referred these auxiliaries to 21, finding that 3 1/2 is 1/6 of 21, 1 1/2 is 1/14 of 21, and 1 is 1/21 of 21, so that he wrote 1/6 1/14 1/21 in black in their proper places (line 3). In this terse manner the scribe obtained his answer to the division,

2 ÷ 7 = 1/6 1/14 1/21.

Still with the same red multiplier 3 and the same reference number 21, we note the 4 (of line 5) was multiplied by 3 giving 12 which was partitioned as 10 1/2 and 1 1/2, each of which divides the reference number 21 in integers 2 and 14, which he wrote as 1/2 and 1/14 in their proper places (line 5) in black. It was thus he obtained his answer to the division,

4 ÷ 7 = 1/2 1/14.

We have no way of telling how the scribe came to choose 3 as his multiplier and, consequently, 21 as his reference number. Nor do we know how he decided upon his particular partitions of 6 and 12."

It could be that Gillings has the order of the procedure backward. The author might have obtained his division in some other way and then supplied the red auxiliaries as an easy check to the procedure. We should note that the red auxiliaries recorded in this table add up to 21, i.e., the reference number, which in this instance is less than a least common denominator where all the red numbers would be integral parts of it, since the auxiliary numbers here include fractions as well as integers. It is also obvious that if the author had halved the product of 4 x 1/7 = 1/2 1/14, he would have found the answer 1/4 1/28 that appeared in the Table of Two for the division of 2 by 7.

See also O. Neugebauer's discussion of this ostracon in *The Exact Sciences in Antiquity* (Princeton, 1952), pp. 87-89. He believes that Hayes' effort to restore the original problem by adding "cubit, palm (?)" at the beginning is "very doubtful." That suggested addition which Neugebauer would reject precedes four fractions: 1/3, 1/14, 1/2, 1/21. I have not discussed them here since they seem to have no relation to the succeeding computation. Neugebauer then goes on to explain the use of the red auxiliary numbers in this computation with the adoption of the red number 1 for 1/21. He suggests that by this stratagem the calculator is abandoning the natural fraction 1/4 in the sequence 1/2, 1/4, ... in favor of 1/3 belonging to the sequence 2/3, 1/3, 1/6, ...

[36] Among the tables that occupy the first half of the Akhmîm Mathematical Papyrus, which reflects Egyptian practices but which is written in Greek and

dates from more than two millennia later in the Byzantine period, say about the seventh or eighth centuries A.D. [see J. Baillet, "Le Papyrus mathématique d'Akhmîm," *Mémoires publiés par les membres de La Mission Archéologique Française au Caire*, Vol. 9 (1892), p. 4], the first part of the table, which gives the multiplication of the first nine units by 1/10, gives the same numbers for the quotients or products in this as the table in the Rhind Papyrus, discussed here except for the following entries (*ibid.*, p. 28): 1/10 x 3 = 1/4 1/20 (instead of its equal 1/5 1/10 in the RMP), 1/10 x 7 = 1/2 1/5 (instead of 2/3 1/30 in the RMP), 1/10 x 8 = 1/2 1/4 1/20 (instead of 2/3 1/10 1/30 in the RMP), and 1/10 x 9 = 1/2 1/3 1/15 (instead of 2/3 1/5 1/30).

[37] See Chace, *op. cit.* in note 4, Vol. 1, p. 100.

[38] See Gillings, *op. cit.* in note 5, pp. 24-25, where he gives instances of the taking of 2/3 of varions quantities from the Rhind Papyrus (Document IV.1).

[39] Peet, *op. cit.* in note 3, p. 20.

[40] *JEA*, Vol.12 (1926), pp. 125-26.

[41] Gillings, *op. cit.* in note 5, Chap. 4, pp. 24-38. The table in Problem 61 of Document IV.1 is crucial to his discussion.

[42] *Ibid.*, pp. 39-40.

[43] Again note that I have changed Gillings' use of the reciprocal form of the unit fractions to the form that writes out the unit fractions with numerator 1 and slant-bar.

[44] *Ibid.*, pp. 40-44.

[45] *Ibid.*, Appendix 11.

[46] *Ibid.*, Chap. 21 and Table 21.1.

[47] T.E. Peet, "Mathematics in Ancient Egypt," *Bulletin of the John Rylands Library Manchester*, Vol. 15 (1931), p. 417. For the verb *sny* or *sni* see R.O. Faulkner, *A Concise Dictionary of Middle Egyptian*, repr. (Oxford, 1972), p. 229. In Document IV.2, Problem 11, Fig. IV.6, Col. XXI, lines 5 and 6 the *n* (—) is missing.

[48] *Op. cit.* in note 47, p. 418.

[49] *Vorlesungen über Geschichte der Mathematik*, Vol. 1 (Leipzig, 1907), pp. 74-76: "An der Spitze dieser Aufgaben stehen die *Hau*-Rechnungen, die dem Inhalte nach nichts anderes sind, als was die heutige Algebra Gleichungen ersten Grades mit einer Unbekannten nennt....Das Wesen einer Gleichung besteht nun allerdings weit weniger in dem Wortlaute als in der Auflösung, und so müssen wir, um die Berechtigung unseres Vergleichs zu prüfen, zusehen, wie Ahmes seine Haurechnungen vollzieht. Er geht dabei ganz methodisch zu Werke, indem er die Glieder, welche, wie man heute sagen würde, links vom Gleichheitszeichen stehen, zunächst in eins vereinigt. Freilich tut er das in doppelter Weise, bald so, dass die Vereinigung im Nebeneinanderschreiben der betreffenden Stammbrüche bestehend nur eine formelle ist, z.B. No. 31.: (1 2/3 1/2 1/7) x = 33; bald so, dass durch Zurückführung auf einen Generalnenner

wirkliche Addition vorgenommen ist, z. B. No. 24.: (8/7) x = 10; No. 28.: (10/9) x = 10; No. 29.: (20/27) x = 10. Im erstgenannten Falle wird sofort durch den Koeffizienten der unbekannten Grösse in die gegebene Zahl dividiert...d. h. bei No. 31. man vervielfältigt 1 2/3 1/2 1/7 solange bis 33 herauskommen und findet...Wert des Haufens 14 1/4 1/97 1/56 1/679 1/776 1/194 1/388.... Der zweite Fall eröffnet wieder zwei Möglichkeiten. Entweder man löst *(a/b)* $x = C$ indem die Division *C/a* vollzogen und deren Quotient mit *b* vervielfacht wird; so in No. 24, wo zuerst 8 in 19 als 2 1/4 1/8 mal enthalten und dann 7 mal 2 1/4 1/8 als 16 1/2 1/8 gefunden wird. Oder aber man dividiert mit *a/b* in 1 und vervielfacht diesen Quotienten mit *C*; so wahrscheinlich in den Aufgaben No. 28 und 29."

[50] Cantor in the next paragraph (on p. 76) suggests the possibility here of the usage of the so-called Hindu method of False Position, as is indeed widely accepted by interpreters of the calculating method involved in the solutions of Problems 24-27. He says that when the intermediate text is missing it is "almost a matter of taste *(Geschmackssache)* whether one will recognize the one or the other [technique as suitable]." But, as our translation and discussion shows, in Problems 24-27 the method of false position was in all likelihood used by the Egyptian author, even though the actual false assumptions were not specified as such and only appear as numbers in the tables of multiplications. Influenced by Chace, I have inserted them as assumptions but only within brackets.

[51] T.E. Peet, "Mathematics in Ancient Egypt," pp. 419-20, as fully cited in note 47.

[52] Here Peet refers to a succeeding footnote (p. 422, n. 2) with the following comments regarding Neugebauer's views: "Nengebauer in a recent publication, *Arithmetik und Rechentechnik der Ägypter (Quellen u. Studien zur Gesch. d. Math[ematik, Astronomie und Physik]*, Abt. B., Bd. 1, Heft 3), pp. 305 ff., denies the use of a trial number, be it unity or any other [number] in all these problems and returns to Cantor's theory that they are solved as equations, in the modern manner, by multiplying the absolute term (on the right in our modern arrangement) by the inverse of the coefficient of x (on the left). The unknown x is, of course, the *ʿḥʿ* or 'quantity', and in one case, M.25 [= MMP.25, i.e., Document IV.2, Problem 25 below], where the equation is $2x + x = 9$, Neugebauer believes that this unknown is explicitly operated on under the name *ʿḥʿ*. The words are 'Add the quantity to the 2; result 3. Divide the 3 into 9; result 3 times. 3 is the number required.' At the same time, even if we accept the curious wording of the text here in spite of the suspicion thrown on it by the occurrence of a vital omission in the setting of the sum (the preposition *ḥnʿ* is followed by no object!), and agree that it involves the explicit use of an unknown and the solution of an equation in the modern style, it does not follow that the same method was used in other cases. Indeed it is by no means a merit

in Neugebauer's hypothesis that it assumes uniformity of treatment in all these problems, even including those which correspond to equations of the second degree, for the outstanding characteristic of Egyptian mathematics is precisely the lack of any such uniformity.

"Thus while I should be sorry to deny outright the possibility of Neugebauer's being right in regard to such problems as are solved by what I have called direct division (R. 30-38, M. 19, 25 [i.e., Document IV.1 (RMP) Problems 30-38 and MMP Problems 19 and 25]), yet I still think that in R. 24-27 [i.e., RMP Problems 24-27], where the coefficient of x (speaking in modern terms) is 1 plus an aliquot part, e.g., 1 1/7 (R. 24), the method used was one of trial, the trial number chosen being, for obvious reasons, in each case the denominator of the aliquot part, e.g., in the case quoted, 7. In this example, if Nengebaner were right, and the process was that of simply dividing the 19 by the coefficient of x, namely 1 1/7, the method would have been that of R. 31-34, namely to operate on 1 + 1/7 to find 19. Yet he asks us to believe that the Egyptian turned 1 1/7 into 8/7 (an improper fraction, be it noted), and then multiplied the 19 by the inversion of this. Where in the papyri can we find a justification for such a procedure? What Egyptian ever took seven-eighths of 19 by dividing it by 8 and multiplying the result by 7....[?]"

[53] Gillings, *op. cit.*, pp. 181-84.

[54] *Science Awakening*, p. 29.

[55] Peet's last words in his "Mathematics in Ancient Egypt," pp. 423-24, seem to reflect a similar belief in the essential absence of algebra in Egyptian mathematics, though to be sure he seems to waffle somewhat in his summary.

[56] Where this number came from is not clear, as I indicate in my text above this note. The table is merely an example of how to find the product of some number and 7 by the usual process of adding the products of doubling the multiples that add up to 7. Hence each term in the following series that starts with 7 and whose multiplier is 7 could be determined in the same way. There have been efforts to link the succeeding series with the old nursery rhyme that starts with the line "As I was going to St. Ives...." but all we have are the simple calculations given here. Chace (*op. cit.*, Vol. 1, p. 30) gets more out of the bare calculations presented when he writes: "Problem 79 is a problem in which is calculated in two ways the sum of a geometrical progression. There are two columns. The first column indicates, it seems to me, a numerical method for determining the sum when the first term is equal to the ratio. This method may be stated in the following rule: In any geometrical progression whose first term is equal to the ratio, the sum of any number of terms is equal to the sum of one less number of terms plus 1 multiplied by the ratio. This rule the author follows in the first column, and in the second he performs the ordinary process of multiplication by the ratio and adds the terms together, getting the same result and thinking perhaps he was proving the rule." In the same volume (p. 112) Chace quotes the nursery rhyme and mentions Rodet's observations concerning the

problem (see below p. 203 n. 109). See also Gillings, *op. cit.*, pp. 166-70, for an extended discussion of Problem 79, where a modern interpretation of the problem is even more pronounced.

57 Neugebauer, *Arithmetik und Rechentechnik* (*cit.* in note 29 above), p. 317.

58 Gillings, *op. cit.*, p. 169.

59 Gillings, *ibid.*, pp. 134-36, shows all of this by the use of modern literal and operational symbols, and remarks in admiration (p.136): "However one looks at this 'round-about' method of solution, it is entirely logical and indeed elegant, whether or not the scribe arrived at it by some algebraic or symbolic thought processes, or by some other means."

60 Peet, "Mathematics in Ancient Egypt," pp. 427-28.

61 L. Borchardt, "Besoldungsverhältnisse von Priestern im mittleren Reich," *ZÄS*, Vol. 40 (1903/1904), pp. 113-17. Borchardt believed that the scribe in fact divided the bread and two beers into 42 portions, though the text lists only 41 2/3. This would have resulted in a jumble of scribal errors in the final column regarding the portions of the second beer. Gillings, *op. cit.*, pp. 124-27 accepted Borchardt's belief of the divisions by 42 and accordingly added a corrected last column on that basis. But if we assume, with Michel Guillemot, Chapter 3 of *Histoire de fractions, fractions d'histoire*, coordonné par P. Benoit *et al.*, (Basel, Boston, and Berlin, 1992), Chap. III, pp. 54-60, that the division was by 41 2/3, as the text seems to say, and that the author accepted more realistic approximations, we have a more sensible interpretation of the text. The reader will notice the wide-spread use of approximations in other account-books (see Document IV.6 below).

62 See the similar remarks on the probable grahic origin of the formula made by Peet, "Mathematics in Ancient Egypt," p. 432: "It seems much more natural, however, to accept the hint of a graphic solution offered by this reference to a rectangle on half the base, and to suppose that the Egyptians had rightly solved the scalene triangle by means of some such figure as Fig. 3 [*Author* = my Fig. IV.35a]. This belief is strengthened by R.52 [=Document IV.1, Problem 52; see the text over the next note], where a truncated triangle is proposed for solution." The reader will find Peet's whole treatment of the area of the triangle (pp. 430-34) of interest.

63 *Ibid.*, pp. 432-33.

64 Gillings, *op. cit.*, p. 140.

65 *Ibid.*, p. 139.

66 *Ibid.*, pp. 143-45. For Michel Guillemot's clever proposal about the nature of the inscribed octagon given in Problem 48 and described briefly in the text above following Gillings' proposal, see his (Guillemot's) "A propos de la 'géométrie égyptienne des figures'," *Sciences et techniques en perspective*, Vol. 21 (1992), pp. 138-40 (full article, pp. 125bis-146). See also Fig. IV.40 below.

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[67] L. Borchardt, "Altägyptische Werkzeichnungen," *ZÄS*, Vol. 34 (1896), pp. 75-76 (entitled "Construction einer Ellipse aus Luqsor"): "Zum Schluss mag hier noch eine weitere Zeichnung Erwähnung finden, wenn dieselbe auch wohl kaum als Werkzeichnung anzüsehen ist.

"Im Luqsortempel, an der Ostwand des östlichen von der späteren koptischen Kirche abgehenden Raumes, befindet sich gegenüber der Thür in Augenhöhe in die Wand gekratzt die Construction eines elliptischen Ovals (s. Taf. VI, Fig. 7 [=my Fig. IV.42]). Als Hülfslinien für die Herstellung desselben sind die Seiten eines liegenden Rechtecks benutzt, dessen Ecken durch symmetrisch angelegte Querlinien abgeschnitten sind.

"Die Construction ist angenähert etwa folgende:

"In dem Rechteck *ABCD*, dessen Seitenlängen $AB = DC = 2a = 2 + 1/2 + 1/4$ Ellen und $AD = BC = 2c = 1 + 2/3$ Ellen (zu je etwa 53 cm) sind, werden auf den Längsseiten von den Ecken aus die Strecken AA_1, BB_1, CC_1 und $DD_1 = 1/4\ AB = a/2$, sowie auf den Schmalseiten die Strecken AA_2, BB_2, CC_2 und $DD_2 = 1/6\ AB = a/3$ abgetragen. Die Mittelpunkte der durch $A_2A_1B_1B_2C_2C_1D_1D_2$ gehenden ovalen Korblinien liegen erstens auf den Mitten der Langseiten in x, x_1 und zweitens auf dem Schnittpunkt der Linien xB_2 und x_1C_2 oder xA_2 und x_1D_2.

"Die Axen der so entstehenden Curve sind angenähert 2 und 3 Ellen, die Mittelpunkte der kleinen Kreisbogen sind angenähert 2 Ellen von einander entfernt. Dies die eine Möglichkeit, die Construction zu erklären.

"Eine zweite Deutung der Zeichnung scheint jedoch auch nicht ausgeschlossen. Ich halte es nämlich nicht für unmöglich, dass wir hier einen Versuch vor uns haben, den Inhalt einer Ellipse mit den Radien 1 und 1 1/2 Elle zu ermitteln, analog der aus dem Londoner mathematischen Papyrus bekannten Aufgabe vom Inhalt des Kreises [$(64/81)\ d^2$ anstatt $(\pi/4)\ d^2$]. In unserer Ellipsen-Aufgabe wäre der Inhalt [$ab\pi = 1 \bullet 1\ 1/2 \bullet \pi = 4.71$ Quadrat-Ellen] etwa gleich dem des Rechtecks [$1\ 2/3 \bullet 2\ 3/4 = 4.58$ Quadrat-Ellen] gesetzt. Der Fehler wäre hierbei nur 13/471, d.h. etwa 1/36. Es wäre auch möglich, dass die Inhaltsermittelung so erfolgt ist: Die Ellipse mit den Durchmessern von 2 und 3 Ellen ist gleich einem Rechteck, dessen Seiten auf jeder Seite 1 Spanne (0.75 m) kürzer sind als die Durchmesser der Ellipse [$(2-2/7) \bullet (3-2/7) = 4.65$ Quadrat-Ellen]. Hierbei wäre der Fehler sogar nur 6/471, d.h. etwa 1/78.

"Was von allen diesen Möglichkeiten das Richtige ist, kann ich wegen der Ungenauigkeit und mangelhaften Erhalten der Zeichnung nicht feststellen.

"Auch die Entstehungszeit ist zweifelhaf; einen terminus post quem giebt uns die Mauer selbst, auf welche die Construction aufgetragen ist. Sie ist nach der Zeit Ramses' III. ausgeführt." [I have converted the commas used in the decimal fractions in this passage to periods for the convenience of the readers with English.]

S. Couchoud, *op. cit.*, p. 66, makes the following observatious concerning this drawing of the ellipse: "Celle-ci [i.e., le dessin] est composée à l'aide de parties de cercles de différents centres et diamètres.

"Ce dessin montre l'approximation d'une ellipse et d'un rectangle qui représente, à une petite erreur près (d'environ 1%), la surface de l'ellipse. Elle peut être définie par analogie avec le cercle par la formule:

$$[a-(1/9)a] \times [b-(1/9)b] = a \times b \times (64/81) \approx a \times b \times (\pi/4)$$

"Que le rectangle ait servi comme aide de construction ou comme représentation pour la surface n'a en fait que peu d'importance: l'essentiel est bien que le dessin donne une preuve nette que l'idée même de l'ellipse n'était pas étrangère à l'esprit égyptien. On pouvait donc la construire et en toute probabilité la calculer. Rappelons de plus que, de son côté, Daressy ["Un tracé égyptien d'une voûte elliptique, "*ASAE*, Vol. 8 (1907) pp. 237-41] a cru pouvoir reconnaître, dans le dessin de construction d'une voûte, l'arc d'une ellipse." We should note Daressy's concluding comments concerning the sketch of what he believed to be an ellipse before the vault in the burial chamber of Ramesses VI: (*Ibid.* p. 241): "Ce simple croquis nous donne donc plusieurs renseignements: il nous fait connaître un des moyens pratiques usités par les Égyptiens pour faciliter le travail des sculpteurs, nous fournit une valeur de la coudée sous Ramsès VI, et enfin nous apprend que mille deux cents ans avant notre ère l'ellipse était connue et employée pour les travaux d'art."

For later building sketches of other curves, see the remarks of Borchardt that immediately precede those on the ellipse in his section enititled "Eine Hohlkehle aus Edfu," *op. cit.* in note 67, pp. 74-75 and Taf. V, Fig.6.

68 See G. Wolff, "Ägyptische Mathematik in Kunst und Handwerk," *Die deutsche höhere Schule*, Heft 13/16 (1941), pp. 266-67. Note that Wolff uses "Handbreiten" where I have used "palms". Handbreit is ambiguous for as I noted in the table I have given in the section on Egyptian Measures earlier in this chapter, a "handsbreadth" (as appearing in Griffith's version of Lepsius' table, but now usually "handbreath") equals 5 fingers or digits, while a palm is 4, and the text on the limestone fragment surely intends the measure to be a "palm" (see Fig. IV.43 below). The difficulty is that now "handbreit" is usually translated into English as "palm". One might be tempted to call this technique of indicating the variation in the ordinate heights of a curve on eqnally spaced horizontal distances an adumbration of analytic geometry. But this should be strongly resisted, for there is no evidence that it represents an effort to draw a geometrical curve whose character has been expressed as a function either algebraically or rhetorically. Perhaps we might call it a kind of early descriptive geometry born of practical measurement, for in fact many building or work sketches that reflect this Egyptian descriptive geometry have been found. See Wolff, "Über die Anfänge der darstellenden Geometrie," *Unterrichtsblätter für Mathematik und Naturwissenschaften*, Vol. 47(1941), pp. 163-68. I shall dis-

cuss this development of Egyptian geometric representation in discussing the nature of Egyptian representations of nature in the next volume.

[69] Peet, "Mathematics in Ancient Egypt," pp. 436-37, remarks on several of these efforts to show how the Egyptians found the correct procedure for finding the volume of a truncated pyramid. See also the multiple references to previous treatments in Gillings, *op. cit.*, pp. 187-93 (which of course includes his own proposals). I have given Gunn-and-Peet's and van der Waerden's suggestions in my succeeding quotations. Incidentally, Peet declares that "Egyptian solid geometry reaches its highest point in M.[Problem] 14," a judgment often echoed in other accounts.

[70] B. Gunn and T. Eric Peet, "Four Geometrical Problems from the Moscow Mathematical Papyrus," *JEA*, Vol. 15 (1929), pp. 179-80.

[71] *Science Awakening*, the English edit. (Groningen, Holland, 1961), pp. 34-35. Note that earlier Quido Vetter, "Problem 14 of the Moscow Mathematical Papyrus," *JEA*, Vol. 19 (1933), pp. 16-18, started with a frustum of the same kind of special pyramid, namely one in which two sides are at right angles to the base and to each other. Furthermore, both authors assume that the Egyptian geometer knew the general formula for the volume of a pyramid, namely $V = (h/3) \times \text{base}$. But Vetter also assumed and specified that, in the special frustum, the side of the upper surface is half the side of the lower, as in Problem 14. Furthermore, instead of transforming the frustum first into three rectangular parallelepipeds and a pyramid followed by the transformation of the pyramid into a fourth parallelepiped, as van der Waerden was to do in the passage I have quoted in the text, Vetter (see Fig. IV.9D) dissects the frustum into two quadrilateral pyramids, *ABCDG* and *EFGHA*, and two trilateral pyramids, *ABFG* and *ADGH*. The sum of these four volumes is precisely equal to the volume of the whole frustum. If a and b are respectively the sides of the lower and upper square bases, then we may express the final volume in algebraic form as

$$V = (h/3)\,a^2 + 2\,(b/3)\,(ah/2) + (h/3)\,b^2$$
$$= (a^2 + ab + b^2)\,h/3,$$

which latter expression conforms, as a general statement, to the Egyptian numerical procedure where $a = 4$, $b = 2$, and $h = 6$.

[72] [See O. Neugebauer, *Vorlesungen über Geschichte der antiken Mathematischen Wissenschaften* (Berlin, 1934), p. 128].

[73] Gunn and Peet, *op. cit.* in note 70, pp. 180-82.

[74] *Ibid.*, pp. 182-83.

[75] "The Truncated Pyramid in Egyptian Mathematics," *JEA*, Vol. 16 (1930), pp. 242-49. Vogel notes (p. 245) that "[assuming that the volume of a pyramid or cone can be worked ont,] the volume of the truncated solid can be determined as the difference of two complete solids without further special formulae, so long as one can first, by means of a proportion, work ont the height of the pyramid or the cone needed to complete it. This method is enounced by Heron in *Metrica* II, 7." He further suggests (p. 249) that the basic formula for the

frustum could be worked out by taking the average of the three areas a^2, ab, and b^2 and multiplying it by the height, that is:

$$V = [(a^2 + ab + b^2) / 3] \bullet h .$$

But of course in Problem 14 of the Moscow Papyrus we saw that the author did not take one-third of the sum of the areas but rather took one-third of the height. Incidentally, regarding the discovery of the formula for the whole pyramid Gillings, *op. cit.*, pp. 189-92, has the following observations:

"It would of course have been a simple operation to construct a hollow pyramid and a hollow rectangular box of the same base and height, to determine that the pyramid had a capacity exactly one-third of the box by simply pouring sand or water. That the Egyptians understood the volume of a rectangular solid to be l x b x d is well attested, so that the volume of an equivalent pyramid would be expressed as one-third of the area of the base times the height, or one-third of the height times the base....Not so simple is the method of dissection, in which a pyramid is cut up and the parts reformed into a rectangular solid whose volume can easily be calculated. None of the dissections I have seen are simple or convincing, but I suggest the following would be within a scribe's capabilities. A right pyramid is constructed of clay or wood, whose perpendicular height is exactly half the side of the square base. This pyramid is then cut into four equal oblique pyramids by two planes passing through the vertex and the midpoints of opposite base lines...[see Fig. IV.9E]. Then three of these four oblique pyramids fit together to form a cube whose sides are half the base of the pyramid. Therefore in volumes the cube is 3/4 the pyramid, or the pyramid is 4/3 the cube. Then the volume of the pyramid is found to be V = (1/3) ha^2." Gillings goes on to present still another another graphic solution, which I omit here.

[76] Vol. 1, *Vorgriechische Mathematik* (Berlin, 1934), pp. 129-37. Neugebauer gives a third interpretation of the problem, namely, as the approximate surface area of a dome-shaped grannary (see Fig. IV.39).

[77] At this point I mention only the second edition of E. Iverson's well known *Canon and Proportions in Egyptian Art* (Warminster, England, 1975), a revision accomplished with the collaboration of Yoshiaki Shibata,

[78] For example, see the discussion and citations of Peet, "Mathematics in Ancient Egypt," pp. 437-41. One of the most careful attempts to avoid anachronism (at least with respect to geometry) is that of M. Guillemot, "A propos de la 'géométrie égyptienne des figures'," *Sciences et techniques en perspective*, Vol. 21 (1992), pp.125bis-146, a paper I have referred to earlier in note 66. He tends to prefer the belief that many of the so-called rules or formulas originate not ont of formal geometrical argument but rather out of satisfying economic needs with skillfully used arithmetic procedures applied to measurements.

Part Two

Documents

DOCUMENT IV.1

The Rhind Mathematical Papyrus: Introduction

I have selected the Rhind Mathematical Papyrus as the first document for two reasons. (1) Though it is not the earliest document (being dated in the introductory passage by its scribe Ahmose as the regnal "year 33, month 4 of [the season] Akhet, [under the majesty of the] King of [Upper] and Lower Egypt Awserre," i.e., the Hyksos King Apophis who reigned c. 1585-1542 B.C.), the scribe goes on to say in its introductory paragraph that it was copied "from an ancient copy made in the time of the King of Upper [and Lower] Egypt, [Nym]atre," i.e., Amenemhet III, who reigned c. 1844-1797 B.C.; hence this puts the work solidly in the classical period of ancient Egyptian mathematics, along with the earliest of the rest of our documents. (2) It reveals the character and extent of Egyptian mathematics better than any of the succeeding documents and so makes an excellent introduction to Egyptian mathematics.

Let us first review briefly the history of the Rhind Mathematical Papyrus. As the facsimile edition of it published by the British Museum notes, it was discovered in a small building near the Ramesseum, which lies on the west bank of the Nile at Thebes, and it was purchased in Luxor in 1858 by A. Henry Rhind, an English traveler and writer.[1] The sale was a part of his purchase of a number of Egyptian antiquities.[2] Following a later visit to Egypt, Rhind died on the way home. His executor sold the papyrus (then in two pieces) to the British Museum in 1865 and these two pieces were given separate numbers: BM 10057 and 10058. The two pieces were once parts of a single roll but were separated in modern times. Further fragments of the original were discovered by P.E. Newberry in 1922. They had been purchased at Luxor in 1862-63 by the American Edwin Smith, whose interest in Egyptian papyri I have already mentioned in Document III.2 of Volume Two of my work. Incidentally, the Smith fragments, which came from the gap between the two pieces purchased by the British Museum, were first housed by the Historical Society of New York (where Newberry found them) but are now at the Brooklyn Museum. Detailed measurements of the pieces were given by Eric Peet, who refers also to earlier discussions.[3] The most recent measurements (diminished from the earlier ones) were occasioned by the conservation work on the papyrus done during recent years; these later measurements are given in a note by T.G.H. James:[4]

> The present measurements for BM 10058 are 199.5cm by 32cm, and for BM 10057, 295.5cm by 32cm; again 18cm are allowed for the gap. The diminution in measurements [from those given by Peet and Chace] is due to the conservation work carried out on the papyrus in recent years. After the document was received into the British Museum its two parts were mounted on a backing card with an unidentified adhesive under some pressure. The deterioration of this card led to splitting and damage (but not loss) to the papyrus. Only during the last ten years have techniques been developed which have allowed the safe removal of long documents from their backing. Using these new techniques conservation staff in the British Museum have been able successfully to detach and remount the two sheets without applying a permanent backing. The removal of the backing and the relaxation of the stresses which had affected the papyrus over more than one hundred years led to the shrinking of the fabric of the sheets. Further reduction in length resulted from the closing of many small gaps where the papyrus and its backing had split. The condition and appearance of the papyrus are now greatly improved.

First and very preliminary accounts of the papyrus and its significance were given by F. Lenormant[5] in 1867, S. Birch[6] in 1868 and H. Brugsch[7] in 1874. The brief paper of the last of these authors was criticized by August Eisenlohr,[8] who in 1877 effected the first publication and translation of the hieratic text.[9] The next important landmark in the history of the papyrus was a series of excellent articles appearing from 1891 through 1894 and written by F.Ll. Griffith, one of the truly great students and translators of the ancient Egyptian language.[10] The next step in the spread of the contents of the Rhind Papyrus was the publication in 1898 of a new facsimile edition by the British Museum, and, as I have noted in notes 9 and 10, it was reviewed in 1899 by Griffith and compared to Eisenlohr's edition, neither one of which was, strictly speaking, a "facsimile." And neither was without considerable fault. In fact, Battiscombe Gunn's negative judgments on both editions, in the course of his 1926 review of Eric Peet's new version of the papyrus, seem more than justified and thus are worthy of quotation here.[11]

> Shortly after their acquisition of the main [Rhind Mathematical] papyrus the Trustees of the British Museum had lithographic plates prepared for a facsimile reproduction; in the publication, however, they were anticipated by August Eisenlohr, who in 1877 brought out facsimile plates accompanying a treatise, *Ein Mathematisches Handbuch...*, which until now has remained the only comprehensive treatment of the document. Not until 1898 did the British Museum facsimile appear, by no means an improvement on its predecessor. (*Plus a note which reads*: While Eisenlohr's plates respect the divisions of the text, one might think that a blind man had been entrusted with the division of the British Museum plates.) Eisenlohr's book, now nearly 50 years old, is both antiquated and unsatisfactory in treatment: not only does it contain a quantity of wrong readings, translations and interpretations, and omits the fragments in America, but also the explanations of the exercises are often complicated and abstruse to a degree which is wholly unnecessary in dealing with a mathematical system so simple in its principles as the Egyptian one. For many years a new edition of

the papyrus has been badly needed; this want is now ably supplied by Professor Peet's work.

As the reader has seen in Chapter Four and will continue to observe in the notes to the translation of the Rhind Papyrus below, the Peet edition of the papyrus published in 1923 proved to be of great importance both for the understanding of this work and the general profile of ancient Egyptian mathematics. But prior to the appearance of Peet's work, indeed even before the publication of the British Museum facsimile edition of the Rhind Papyrus, a large number of articles and books of varying significance for understanding the papyrus were published. We can note as very important studies, for example, F. Hultsch's 1895 treatment of Egyptian *Theilungsrechnung*[12] and K. Sethe's superb monograph: *Von Zahlen und Zahlworten bei den alten Ägyptern und was für andere Völker und Sprachen daraus zu lernen ist. Ein Beitrag zur Geschichte von Rechenkunst und Sprache* (Strasbourg, 1916). In addition to the somewhat later edition of Chace (see above, note 1), which is still by far the best version of the Rhind Papyrus and which will be mentioned again and again in the course of presenting this document, a whole host of other authors of more recent vintage (like Neugebauer, Van der Waerden, Bruins, Gillings, Couchoud, and Guillemot, to specify a few whose works are listed in my bibliography below) will be mentioned in the notes or in Chapter IV when I consider their views to be pertinent.

Though I have discussed the Rhind Papyrus at some length in Chapter Four above and certainly do not wish to repeat that discussion here, it seemed appropriate to include here a brief outline of the chief subjects of the papyrus in the order of their appearance:

Table of the Division of 2 by the Odd-numbers 3-101 (Chace's Plates 2-33);

Table of the Divisions of the Numbers 1-9 by 10 (*ibid.*, Plate 33);

Problems 1-6: the Successive Divisions of 1, 2, 6, 7, 8, and 9 Loaves of Bread among 10 Men (*ibid.*, Plates 34-38);

Problems 7-20: Multiplication of some Fractional Expressions by other Expressions that include the number 1 and a sum of Fractions that consists only of Unit Fractions (Problems 7-15, *ibid.*, Plates 39-42) or by still other Expressions that include the number 1 and a sum consisting of 2/3 plus some Unit Fractions (Problems 16-20, *ibid.*, Plates 42-43);

Problems 21-23: Completion of given sums of Unit Fractions to make 1 in the first two problems (*ibid.*, Plates 44-45), or to make 2/3 in the third problem (*ibid.*, Plate 46);

Problems 24-29: Quantity (*ʿḥʿ*) Problems, i.e., Problems of Finding an Unknown Quantity when an expression involving the Unknown and Fractions of it is specified (*ibid.*, Plates 47-51);

Problems 30-34: Problems of determining an Unknown Quantity when an expression involving the addition of the Unknown and Fractional Parts of it is specified (*ibid.*, Plates 52-56);

Problems 35-38: Divisions of a Heqat-Measure (*ibid.*, Plates 57-60);

Problems 39-40: Divisions of Loaves involving Arithmetic; Progression (*ibid.*, Plates 61-62);

Problems 41-46: Finding the Volumes of Three Cylindrical Granaries and One Rectangular Granary (*ibid.*, Plates 63-68);

Problem 47: Division of 100 Heqat (*ibid.*, Plate 69);

Problems 48-55: Area Problems, including a Circle, its Circumscribing Square and its equivalent square, Triangles, Trapezoids, and other Setjat problems (*ibid.*, Plates 70-77);

Problems 56-60: Pyramids and the Relations of the Lengths of Two Sides of a Triangle, i.e., those involving Seqed (Slope) and Altitude Determinations (*ibid.*, Plates 78-82);

Problems 61-84: Miscellaneous Arithmetical Determinations (*ibid.*, Plates 83-105).

The remaining bits of the papyrus (Numbered 85-87) do not appear to belong to the mathematical tract that precedes them. Specific comments of mine and others on various aspects and problems of the mathematical tract are included in the endnotes that follow the translation.

In presenting my English translation of the famous Rhind Papyrus I have followed the hieratic text of A.B. Chace, given here with hieroglyphic transcriptions in Figs. IV.2a-IV.2aaa, and I have also stayed close to Chace's translation. In doing so, I have silently accepted most of his decisions concerning the false appearance or lack of appearance of the dots indicating unit fractions. I have also accepted his standardizing of the check marks or strokes that indicate the lines to be added in the various parts of the Table of Two and the subsequent problems. The reader may easily locate, and will benefit from, Chace's explanatory additions to the translation in Volume One and his various comments concerning the state of the text in Volume Two of the work.

One distinction of my version from that of the earlier authors is that I have bracketed all additions I have made in order to clarify the rather succinct and syncopated Egyptian text. For instance, in the Table of Two I have bracketed and italicized each of the successive divisions of two by the odd numbers from 3 to 101, thereby making these bracketed and italicized phrases subtitles for the separate solutions. Some of my notes refer to these bracketed additions, while many others attempt to clarify further the aims of the author in his mathematical operations. Still others discuss possible alternate translations of the Egyptian words used by the author.

The use of bold-faced type in the translation indicates rubrication appearing in the papyrus. The categories of words, phrases, and numbers rubricated by the scribe are the following: (1) the general title of the work at the beginning of it, (2) the word "**Call**" (*nyś*) or the full expression "**Call 2**" with which the first fractional division of 2 on each page of the Table of 2 begins (and which is to be understood as applying to every division of 2 on that page of the table; see the divisions by 3, 17, 29, 41, 53, 65, 77, 89, and 101), (3) the initial numbers in each division of two that are the parts of (i.e., the fractional multipliers of) the denominators that effect products which add up to 2, (4) the word "**Procedure**" (*śśmt*), which appears before the working out of the division in many cases,

(5) the so-called red auxiliary numbers used in the Egyptian version of common denominators that are often put under the unit fractions that are being added in the problems that follow the Table of Two, and (6) the slant checks (\ ) before each of the lines of multiplications that are to be added (but rubricated in the Table of Two only through the division of 2 by 9). All of these usages can be most easily found in Chace's plates, which are included in Figs. IV.2a-IV.2aaa. But they can also be found, with somewhat more difficulty because of the complexity of the actual layout on each page of the papyrus, in the recent photographic plates of the work of Robbins and Shute (see note 2). I have not converted the rubricated check marks to bold-faced type, but have done so, for the most part, in the cases of other usages. I note further that I have represented the Horus-eye fractional signs for the following fractions of a heqat: *1/2, 1/4, 1/8, 1/16, 1/32,* and *1/64*, when they appear in the Rhind Papyrus and elsewhere, by Italic type (although Chace in his translation of the Rhind Papyrus uses bold-faced type). I do this since I have already usurped boldface for rubrication.

Finally I must explain why I have adopted the somewhat ambiguous form of writing fractions with the numbers all on the same line (e.g., 1/2, 2/3, etc.). The "equation program" included in Microsoft Word, Version 6, by which fractions such as $\frac{1}{2}, \frac{2}{3}, {}^{1}\!/_{2}, {}^{2}\!/_{3}$ may be written, is rather clumsy to use when there are a great many fractions to be inserted in a normal expository text, or in a text that is partly tabular and partly expository. For the same reason I have not adopted the reciprocal form of writing unit fractions with a bar over the denominator that replaces the unit numerator, which has become so popular since its use by Neugebauer. Hence I have fallen back on the old-fashioned forms mentioned above. The ambiguity is reduced by the insertion of one or more spaces after each fraction. I also must remind the reader that there is no operational symbol for addition in the papyrus, and hence when the text has a string of numbers and unit fractions when working out the steps to solve each division of 2 by an odd number from 3 to 101, I have merely put a space between successively added numbers and their fractions, as for example when giving the table for solving the division of 2 by 13, I give the result of multiplying 13 by 1/8 as 1 1/2 1/8 (as it is in the papyrus) rather than as 1+1/2+1/8 or 1 1/2 + 1/8. This should not cause any difficulty to the attentive reader, since almost all of the calculations are immediately obvious.

Notes to the Introduction to Document IV.1

[1] See *Facsimile of the Rhind Mathematical Papyrus* (London, 1898), Preface. Cf. T. Eric Peet, *The Rhind Mathematical Papyrus: British Museum 10057 and 10058* (London, 1923), p. 2, and A.B. Chace et al., *The Rhind Mathematical Papyrus*, Vol. 1 (Oberlin, Ohio, 1927), p. 1.
[2] G. Robins and C. Shute, *The Rhind Mathematical Papyrus: an Ancient Egyptian Text* (London, 1987).
[3] Peet, *op. cit.* in note 1 above, pp. 2-3. Cf. Chace et al., Vol. 1, pp. 2-3.
[4] Robins and Shute, *op. cit.* in note 2 above, p. 6. See also the general description by Robins and Shute on pp. 10-11.
[5] F. Lenormant, "Note relative à un papyrus égyptien contenant un fragment d'un traité de géométrie appliquée à l'arpentage," *Comptes rendus...de l'Académie des Sciences*, Vol. 65 (1867), p. 903. Given in its entirety in R. Archibald's bibliography published in Vol. 1 of the Chace edition of the Rhind Papyrus, p. 135. The note is quite brief and general. It speaks ouly of the section on the determination of areas and the volume of a pyramid (i.e., the geometric problems later numbered 48-60).
[6] Birch, "Geometric Papyrus," *ZÄS*, Vol. 6 (1868), pp. 108-10. Birch's description is a much better estimate of the papyrus' contents. Thus he says: "It is a treatise on geometry, mensuration and arithmetic combined, the geometric problems being treated arithmetically and not abstractly as by the geometricians of the Alexandrian school.... The Rhind Papyrus...contains a series of propositions relative to values or quantities as they may be called treated arithmetically, each case being a proposition considered separately, the dimension of each square, circle, triangle or pyramid to be copied being given separately and the area or contents superficial or solid thereby calculated, the object being to determine the values or quantities. For the value of fields the author of the treatise uses the isosceles triangle and the trapezoids into which it is susceptible of division by drawing parallel lines to the base. But besides the resolution of

geometric problems, others of a nature more purely arithmetical are also given so that the treatise in reality is that of applied arithmetic. There is nothing in the whole treatise at all like the abstract geometry of the Alexandrian School."

[7] H.K. Brugsch, "Über den mathematischen Papyrus im britischen Museum zu London," *ZÄS*, Vol. 12 (1874), pp. 147-49.

[8] A. Eisenlohr, "Berichtung," *ZÄS*, Vol. 13 (1875), pp. 26-29.

[9] *Ein mathematisches Handbuch der alten Ägypter (Papyrus Rhind des British Museum) übersetzt und erklärt* (Leipzig, 1877). Vol. 1 is a translation and commentary; Vol. 2 consists of *Tafeln*, i.e., plates, of the hieratic text from a so-called "facsimile" supplied to him by the British Museum. For the tangled history of this edition and its comparison to the facsimile edition published by the Museum in 1898 (see note 1 above) consult the review by F.Ll. Griffith in *Orientalistische Litteratur-Zeitung*, Vol. 2 (1899), cols. 116-17.

[10] Griffith, "The Rhind Mathematical Papyrus," *PSBA*, Vol. 13 (1891), pp. 328-32; Vol. 14 (1891), pp. 26-31; Vol. 16 (1894), pp. 164-73, 201-08, and 230-48. Consult also Griffith's "Notes on Egyptian Weights and Measures," *PSBA*, Vol. 14, 1892, pp. 403-50; Vol. 15 (1893), pp. 301-15. The relevance of the latter study for the Rhind Papyrus arises from the fact that there are 44 problems in the papyrus in which weights and measures are referred to, and, as Archibald mentions in his bibliography (in Volume I of Chace's edition specified in note 1, p. 154), Griffith considers all of these. Finally, see Griffith's review (mentioned in the preceding note) of the "facsimile" editions of the Rhind Papyrus published by Eisenlohr and the British Museum.

[11] *JEA*, Vol. 12 (1926), p. 123, full review pp. 123-37.

[12] Hultsch, "Die Elemente der agyptischen Theilungsrechnung. Erste Abhandlung," *Abhandlung der philologisch-historischen Classe der königlich-sächsischen Gesellschaft der Wissenschaft*, Vol. 17, no. 1 (1895), 192 pp. The second part was never published, though another paper appeared in 1901: F. Hultsch, "Neue Beiträge zur ägyptischen Teilungsrechnung," *Bibliotheca Mathematica*, Series 3, Vol. 2 (1901), pp. 177-84.

DOCUMENT IV.1

The Rhind Mathematical Papyrus

Accurate reckoning [or Rules for reckoning, i.e.,] for inquiring into things, and the knowledge of all things, mysteries....all secrets.[1] This book was copied in regnal year 33, month 4 of Akhet, [under the majesty of the] King of [Upper and] Lower Egypt, Awserre[2], given life, from an ancient copy made in the time of the King of Upper [and Lower] Egypt, [Nym]atre.[3] The scribe Ahmose writes this copy.

[TABLE OF DIVISIONS OF TWO BY THE ODD NUMBERS 3-101]

[2 divided by 3][4]
Call 2 out of 3 [i.e., Get 2 by operating on 3].[5] **2/3** [of 3 is] 2.

[2 divided by] 5
1/3 [of 5 is] 1 2/3, **1/15** [of 5 is] 1/3.
Procedure [*ššmt*, i.e., Working Out]:[6]

	1	5
	2/3	3 1/3
\	1/3	1 2/3
\	1/15	1/3.

[2 divided by 7][7]
1/4 [of 7 is] 1 1/2 1/4, **1/28** [of 7 is] 1/4.

	1	7			
	1/2	3 1/2		1	7
\	1/4	1 1/2 1/4		2	14
\	4	28	1/4	4	28.

[2 divided by 9]
1/6 [of 9 is] 1 1/2, **1/18** [of 9 is] 1/2.

1	9	
2/3	6	
1/3	3	
\ 1/6	1 1/2	
\ [2	1]8	1/2.

[2 divided by 11]
1/6 [of 11 is] 1 2/3 1/6, **1/66** [of 11 is] 1/6.

[1	11				
2/3	7 1/3		[1]	11	
1/3	3 2/3		[\ 2	2]2	
\ 1/6	1 2/3	1/6]	[\ 4	4]4	
		[Total]	6	66	1/6.

[2 divided by 13]
1/8 [of 13 is] 1 1/2 1/8, **1/52** [of 13 is] 1/4, **1/104** [of 13 is] 1/8.

1	1[3]	
1/2	6 1/2	
1/4	3 1/4	
\ 1/8	1 1/2 1/8	
\ 4	52	1/4
\ 8	104	1/8.

[2 divided by 15]
1/10 [of 15 is] 1 1/2, **1/30** [of 15 is] 1/2.

1	15
\ 1/10	1 1/2
\ 1/30	1/2.

[2 divided by 17]
Call 2 out of 17 [i.e., Get 2 by operating on 17].
1/12 [of 17 is] 1 1/3 1/12, **1/51** [of 17 is] 1/3, **1/68** [of 17 is] 1/4.

Procedure:

1	17				
2/3	11 1/3				
1/3	5 2/3		\1	17	
1/6	2 1/2 1/3		\2	34	
\1/12	1 1/4 1/6	[Total:][8]	3	51	1/3
Remainder[9]	1/3 1/4		4	68	1/4.

[2 divided by 19]

1/12 [of 19 is] 1 1/2 1/12, **1/76** [of 19 is]1/4, **1/114** [of 19 is] 1/6

1	19				
2/3	12 2/3		1	19	
1/3	6 1/3		2	38	
1/6	3 1/6		4	76	1/4
\1/12	1 1/2 1/12	Remainder	1/6		
Remainder	1/4 1/6				
			1	19	
			\2	38	
			\4	76	
		Total	6	114	1/6.

[2 divided by 21]

1/14 [of 21 is] 1 1/2, **1/42** [of 21 is] 1/2.

1	21	
\2/3	14	1 1/2
\2	42	1/2.

[2 divided by 23]

1/12 [of 23 is] 1 2/3 1/4, **1/276** [of 23 is] 1/12.

1	23				
2/3	15 1/3		1	23	
1/3	7 2/3		\10	230	
1/6	3 1/2 1/3		\2	46	
\1/12	1 1/2 1/4 1/6	Total	[12]	276	1/12.
Remainder	1/12				

[2 divided by 25]
1/15 [of 25 is] 1 2/3, **1/75** [of 25 is] 1/3.

1	25	
\ 1/15	1 2/3	
\ 3	75	1/3.

[2 divided by 27]
1/18 [of 27 is] 1 1/2, **1/54** [of 27 is] 1/2.

1	27	
\ 2/3	18	1 1/2
\ 2	54	1/2.

[2 divided by 29]
Call 2 out of 29 [i.e., Get 2 by operating on 29].
1/24 [of 29 is] 1 1/6 1/24, **1/58** [of 29 is] 1/2, **1/174** [of 29 is] 1/6, **1/232** [of 29 is] 1/8.
Procedure[10]:

1	[29]	
\ 1/24	1 1/6 1/24	
\ 2	58	1/2
\ 6	174	1/6
\ 8	232	1/8.

[2 divided by] 31
1/20 [of 31 is] 1 1/2 1/20, **1/124** [of 31 is] 1/4, **1/155** [of 31 is] 1/5.

1	[31]	
\ 1/20	1 1/2 1/20	
\ 4	124	1/4
\ 5	155	1/5.

[2 divided] 33
1/22 [of 33 is] 1 1/2, **1/66** [of 33 is] 1/2.

[1 33] *[cont.]*

\ 2/3	22	1 1/2
\ 2	66	1/2.

[2 divided by] 35

1/30 [of 35 is] 1 1/6, **1/42** [of 35 is] 2/3 1/6.
6 7 5[11]

[2 times 1/35 is 1/30 1/42. For 1/35 applied to 210 gives 6; and 2 times 6 is 12, or 7 and 5, which are 1/30 and 1/42 of 210.]

[1	35]	
\ 1/30	1 1/6	
\ 1/42	2/3 1/6.	

[2 divided by 37]

1/24 [of 37 is] 1 1/2 1/24, **1/111** [of 37 is] 1/3, **1/296** [of 37 is] 1/8.

1	37				
2/3	24 2/3		\ 1	37	
1/3	12 1/3		\ 2	74	
1/6	6 1/6	Total	3	111	1/3
1/12	3 1/12	Remainder	1/8		
\ 1/24	1 1/2 1/24		1	37	
Remainder	1/3 1/8		2	74	
			4	148	
			8	296	1/8.

[2 divided by] 39

1/26 [of 39 is] 1 1/2, **1/78** [of 39 is] 1/2.

[1	39]	
\ 2/3	26	1 1/2
\ 2	78	1/2.

[2 divided by 41]

Call 2 out of 41 [i.e., Get 2 by operating on 41].

1/24 [of 41 is] 1 2/3 1/24, **1/246** [of 41 is] 1/6, **1/328** [of 41 is] 1/8.

Procedure:

[1	41]				
2/3	27 1/3		1	41	
1/3	13 2/3		\2	82	
1/6	6 2/3 1/6		\4	164	
1/12	3 1/3 1/12	Total:	6	246	1/6
\ 1/24	1 2/3 1/24		8	328	1/8.
Remainder	1/6 1/8				

[2 divided by] 43

1/42 [of 43 is] 1 1/42, **1/86** [of 43 is] 1/2, **1/129** [of 43] is 1/3, **1/301** [of 43 is] 1/7.

	[1	43]	
Find *(gm)*	\ 1/42	1 1/42	
	\2	86	1/2
	\3	129	1/3
	\7	301	1/7.

[2 divided by] 45

1/30 [of 45 is] 1 1/2, **1/90** [of 45] is 1/2.

[1	45]	
\2/3	30	1 1/2
\2	90	1/2.

[2 divided by] 47

1/30 [of 47 is] 1 1/2 1/15, **1/141** [of 47 is] 1/3, **1/470** [of 47 is] 1/10.

	[1	47]	
Find	1/30	1 1/2 1/15	
	\3	141	1/3
	\10	470	1/10.

[2 divided by] 49 [cont.]

1/28 [of 49 is] 1 1/2 1/4, **1/196** [of 49 is] 1/4.

	[1	49]	
Find	\ 1/28	1 1/2 1/4	
	\ 4	196	1/4.

[2 divided by] 51

1/34 [of 51 is] 1 1/2, **1/102** [of 51 is 1/2].

[1	51]	
\ 2/3	34	1 1/2
\ 2	102	1/2.

[2 divided by 53]

Call 2 out 53.

1/30 [of 53 is] 1 2/3 1/10, **1/318** [of 53 is] 1/6, **1/795** [of 53 is] 1/15.

Procedure:

	[1	53]					
Find	\ 1/30	1 2/3 1/10			1	53	
	\ 6	318	1/6		\ 10	530	
Remainder		1/15			\ 5	265	
				Total	15	795	1/15.

[2 divided by] 55

1/30 [of 55 is] 1 2/3 1/6, **1/330** [of 55 is] 1/6.

	[1	55]	
Find	\ 1/30	1 2/3 1/6	
	\ 6	330	1/6.

[2 divided by] 57

1/38 [of 57 is] 1 1/2, **1/114** [of 57 is] 1/2.

[1	57]	
\ 2/3	38	1 1/2
\ 2	114	1/2.

[2 divided by] 59

1/36 [of 59 is] 1 1/2 1/12 1/18, **1/236** [of 59 is] 1/4, **1/531** [of 59 is] 1/9.

	[1	59]	
Find	\ 1/36	1 1/2 1/12 1/18	
	\ 4	236	1/4
	\ 9	531	1/9.

[2 divided by] 61

1/40 [of 61 is] 1 1/2 1/40, **1/244** [of 61 is] 1/4, **1/488** [of 61 is] 1/8, **1/610** [of 61 is] 1/10.

	[1	61]	
Find	\ 1/40	1 1/2 1/40	
	\ 4	244	1/4
	\ 8	488	1/8
	\ 10	610	1/10.

[2 divided by] 63

1/42 [of 63 is] 1 1/2, **1/126** [of 63 is] 1/2.

[1	63]	
\ 2/3	42	1 1/2
\ 2	126	1/2.

[2 divided by 65]

Call 2 out of 65.

1/39 [of 65 is] 1 2/3, **1/195** [of 65 is] 1/3.

Procedure:

	[1	65]	
Find	\ 1/39	1 2/3	
	\ 3	195	1/3.

[2 divided by] 67

1/40 [of 67 is] 1 1/2 1/8 1/20, **1/335** [of 67 is] 1/5, **1/536** [of 67 is] 1/8.

	[1	67]
Find	\ 1/40	1 1/2 1/8 1/20 *[cont.]*

\ 5	335	1/5
\ 8	536	1/8.

[2 divided by] 69

1/46 [of 69 is] 1 1/2, **1/138** [of 69 is] 1/2.

[1	69]	
\ 2/3	46	1 1/2
\ 2	138	1/2.

[2 divided by] 71

1/40 [of 71 is] 1 1/2 1/4 1/40, **1/568** [of 71 is] 1/8, **1/710** [of 71 is] 1/10.

[1	71]	
Find \ 1/40	1 1/2 1/4 1/40	
\ 8	568	1/8
\ 10	710	1/10.

[2 divided by] 73

1/60 [of 73 is] 1 1/6 1/20, **1/219** [of 73 is] 1/3, **1/292** [of 73 is] 1/4, **1/365** [of 73 is] 1/5.

[1	73]	
Find \ 1/60	1 1/6 1/20	
\ 3	219	1/3
\ 4	292	1/4
\ 5	365	1/5.

[2 divided by] 75

1/50 [of 75 is] 1 1/2, **1/150** [of 75 is] 1/2.

[1	75]	
\ 2/3	50	1 1/2
\ 2	150	1/2.

[2 divided by 77]

Call 2 out of 77.

1/44 [of 77 is] 1 1/2 1/4, **1/308** [of 77] is 1/4.

Procedure:

```
        [1          77]
Find  \ 1/44         1 1/2 1/4
      \ 4      3[08]            1/4.
```

[2 divided by] 79

1/60 [of 79 is] 1 1/4 1/15, **1/237** [of 79 is] 1/3, **1/316** [of 79 is] 1/4, **1/790** [of 79 is] 1/10.

```
        [1          79]
Find  \ 1/60       1 1/4 1/15
      \ 3      237              1/3
      \ 4      316              1/4
      \ 10     790              1/10.
```

[2 divided by] 81

1/54 [of 81 is] 1 1/2, **1/162** [of 81 is] 1/2.

```
        [1          81]
      \ 2/3     54              1 1/2
      \ 2      162                1/2.
```

[2 divided by] 63 (! but should be 83)

1/60 [of 83 is] 1 1/3 1/20, **1/332** [of 83 is] 1/4, **1/415** [of 83 is] 1/5, **1/498** [of 83 is] 1/6.

```
       [1       83]
Find \ 1/60      1 1/3 1/20
     \ 4      332             1/4
     \ 5      415             1/5
     \ 6      498             1/6.
```

[2 divided by] 85

1/51 [of 85 is] 1 2/3, **1/255** [of 85 is] 1/3.

```
       1        85
Find \ 1/51      1 2/3
     \ 3      255             1/3.
```

[2 divided by 87]
1/58 [of 87 is] 1 1/2, **1/174** [of 87 is] 1/2.

1	87	
\ 2/3	58	[1] 1/2
\ 2	174	1/2.

[2 divided by 89][12]
Call 2 out of 89.
1/60 [of 89 is] [1 1/3] 1/10 1/20, **1/356** [of 89 is] 1/4, **1/534** [of 89 is] 1/6, **1/890** [of 89 is] 1/10.
Procedure:

[1	89]	
Find \ 1/60	1 1/3 1/10 1/20	
\ 4	356	1/4
\ 6	534	1/6
\ 10	890	1/10.

[2 divided by] 91
1/70 [of 91 is] 1 1/5 1/10, **1/130** [of 91 is] 2/3 1/30.

[1	91]
Find \ 1/70	1 1/5 1/10
Find \ 1/130	2/3 1/30.

[2 divided by] [9]3
1/62 [of 93 is] 1 1/2, **1/186** [of 93 is] 1/2.

[1	93]	
\ 2/3	62	1 1/2
\ 2	186	1/2.

[2 divided by] 95
1/60 [of 95 is] 1 1/2 1/12, **1/380** [of 95 is] 1/4, **1/570** [of 95 is] 1/6.

[1	95]	
Find \ 1/60	1 1/2 1/12	
\ 4	380	1/4

\6 570 1/6.

[2 divided by] 97

1/56 [of 97 is] 1 1/2 1/8 1/14 1/28, **1/679** [of 97 is] 1/7, **1/776** [of 97 is] 1/8.

	[1	97]	
Find	\ 1/56	1 1/2 1/8 1/14 1/28	
	\ 7	679	1/7
	\ 8	776	1/8.

[2 divided by] 99

1/66 [of 99 is] 1 1/2, **1/198** [of 99 is] 1/2.

	[1	99]	
Find	\ 2/3	66	1 1/2
	\ 2	198	1/2.

[2 divided by 101]

[**Call** 2 out of 101.]

1/101 [of 101 is] 1, **1/202** [of 101 is] 1/2, **1/303** [of 101 is] 1/3, **1/606** [of 101 is] 1/6.

Procedure:

[\ 1	101	1]
\ 2	202	1/2
\ 3	303	1/3
\ 6	606	1/6.

[TABLE OF DIVISION BY 10 AND PROBLEMS 1-6]

[Table of Division by 10][13]

[1 divided by 10 yields] 1/10
[2 divided by 10 yields] 1/5
[3 divided by 10 yields] 1/5 1/10
[4 divided by 10 yields] 1/3 1/15
[5 divided by 10 yields] 1/2
[6 divided by 10 yields] 1/2 1/10 *[cont.]*

[7 divided by 10 yields] 2/3 1/30
[8 divided by 10 yields] 2/3 1/10 1/30
[9 divided by 10 yields] 2/3 1/5 1/30.

[Problems 1-6: Divisions of Loaves of Bread among 10 Men]

[Problem 1]

Example of Dividing (*tp n psš*)[14] 1 loaf (*tꜥ*) among 10 men. Do the multiplication (*irḫr*[*k*] *wꜥḥ*-[*tp*]) of 1/10 times 10 [to get 1 loaf].

The procedure [*lit.* doing (*irt*)] is as follows:

[1 1/10
\2 1/5
4 1/3 1/15]
\8 2/3 1/10 1/30

Total: 1 [loaf], which is the same [i.e., the correct total for 10 men after each man receives 1/10 of it].

[Problem 2]

Dividing [2] loaves among [10 men]. Do the multiplication [of 1/5 times 10].

The procedure is as follows: (*irt my ḫpr*):

[1 1/5]
\2 1/3 1/15
[4] 2/3 1/10 1/30
\8 1 1/3 1/5 1/15

Total: 2 [loaves], which is [the correct total for 10 men after each man receives 1/5 of the total].

[Problem 3]

Dividing 6 loaves among [10] men. Do the multiplication of [1/2] 1/10 times 10.

The procedure is as follows: [1 1/2] 1/10
[\2 1] 1/5
[4 2] 1/3 1/15
\8 4 [2/3 1/10] 1/30

Total: 6 [loaves], which is [the correct total for 10 men after each man receives 1/2 1/10 of it].

[Problem 4] [15]

Dividing 7 loaves among 10 men. Do the multiplication of 2/3 1/30 times 10; the result is 7.

The procedure is as follows:	[1]	2/3 1/[30]
	\ [2	1] 1/3 1/15
	4	2 2/3 1/10 1/30
	\ 8	5 1/2 1/10

Total: 7 loaves, which is [the correct total for 10 men after each man receives 2/3 1/30 of it].

[Problem 5]

Divide 8 loaves among 10 men. Do the multiplication of 2/3 1/10 1/30 times 10; the result is 8.

[The procedure is] as follows:	1	2/3 1/10 1/30
	\ 2	1 1/2 [1/10
	4	3 1/5]
	\ 8	6 1/3 1/15
	Total:	8 loaves, which is correct.

[Problem 6]

Divide 9 loaves among 10 men. Do the multiplication of 2/3 1/5 1/30 times 10.

The procedure is as follows:	1	2/3 1/5 1/30
	\ 2	1 2/3 1/10 1/30
	4	3 1/2 1/10
	\ 8	7 1/5
	Total:	9 loaves, which is correct.

[PROBLEMS 7-20: MULTIPLICATION OF CERTAIN FRACTIONAL EXPRESSIONS][16]

[Problem 7][cont.]

[Multiply 1/4 1/28 by 1 1/2 1/4.]
Example of completion (*tp n skmt*):

1	1/4 1/28 [as parts of 28 these are] 7 **[and] 1**
1/2	1/8 1/56 [as parts of 28 these are] 3 1/2 **[and] 1/2**
1/4	1/16 1/112 [as parts of 28 these are] 1 1/2 1/4 **[and] 1/4**
Total:	1/2 [since as a part of 28 this is 14].

[Problem 7B]
[Multiply 1/4 1/28 by 1 1/2 1/4.]

1	1/4 1/ 28
1/2	1/8 1/56
1/4	1/16 1/112 [as parts of 28 these are] 1 1/2 1/4 **[and] 1/4**
Total:	1/2.

[Problem 8]
[Multiply 1/4 by 1 2/3 1/3.]

1	1/4 [as a part of 18 this is] 4 1/2
2/3	1/6 [as a part of 18 this is] 3
1/3	1/12 [as a part of 18 this is] 1 1/2
Total:	1/2 [since as a part of 18 this is] 9.

[Problem 9]
[Multiply 1/2 1/14 by 1 1/2 1/4.]

1	1/2 1/14
1/2	1/4 1/28
1/4	1/8 1/56
Total:	1.

[Problem 10][17]
[Multiply 1/4 1/28 by 1 1/2 1/4.]

1	1/4 1/28
1/2	1/7
1/4	1/14

Total: 1/2.

[Problem 11]

[Multiply 1/7 by 1 1/2 1/4.]

1	1/7
1/2	1/14
1/4	1/28
Total:	1/4.

[Problem 12]

[Multiply 1/14 by 1 1/2 1/4.]

1	1/14
1/2	1/28
1/4	1/56
Total:	1/8.

[Problem 13]

[Multiply 1/16 1/112 by 1 1/2 1/4.]

1	1/16 1/112 [as parts of 28 these are] **1 1/2 1/4 [and] 1/4**
1/2	1/32 1/224 [as parts of 28 these are] **1/2 1/4 1/8 [and] 1/8**
1/4	1/64 1/448 [as parts of 28 these are] **1/4 1/8 1/16 [and] 1/16**
Total:	1/8 [since as part of 28 this is 3 1/2].

[Problem 14]

[Multiply 1/28 by 1 1/2 1/4.]

1	1/28 [as a part of 28 this is] **1**
1/2	1/56 [as a part of 28 this is] **1/2**
1/4	1/112 [as a part of 28 this is] **1/4**
Total:	1/16 [since as a part of 28 this is 1 1/2 1/4].

[Problem 15][18]

[Multiply 1/32 1/224 by 1 1/2 1/4.] *[cont.]*

1	1/32 1/224 [as parts of 28 these are] 1/2 1/4 1/8 [and] 1/8
1/2	1/64 1/448 [as parts of 28 these are] 1/4 1/8 1/16 [and] 1/16
1/4	1/128 1/896 [as parts of 28 these are] 1/8 1/16 1/32 [and] 1/32
Total:	1/16 [since as a part of 28 this is 1 1/2 1/4].

[Problem 16]

[Multiply 1/2 by 1 2/3 1/3.]

1	1/2
2/3	1/3
1/3	1/6
Total:	1.

[Problem 17]

[Multiply 1/3 by 1 2/3 1/3.]

1	1/3
2/3	1/6 1/18
1/3	1/9
Total:	2/3.

[Problem 18]

[Multiply 1/6 by 1 2/3 1/3.]

1	1/6
2/3	1/9
1/3	1/18
Total:	1/3.

[Problem 19]

[Multiply 1/12 by 1 2/3 1/3.]

1	1/12 [as a part of 18 this is] 1 1/2
2/3	1/18 [as a part of 18 this is] 1
1/3	1/36 [as a part of 18 this is] 1/2
Total:	1/6 [since as a part of 18 this is 3].

[Problem 20]
[Multiply 1/24 by 1 2/3 1/3.]

1	1/24 [as a part of 18 this is] 1/2 1/4
2/3	1/36 [as a part of 18 this is] 1/2
1/3	1/72 [as a part of 18 this is] 1/4
Total:	1/12 [since as a part of 18 this is 1 1/2].

[PROBLEMS 21-23: PROBLEMS IN COMPLETION]

[Problem 21]
It is said to you: Complete 2/3 1/15 to 1.
[Applied to 15, 2/3 is] 10 [and 1/15 is] 1.
The total is 11 and the remainder is 4.
Multiply 15 in order to find (i.e., get) 4.

1	15
1/10	1 1/2
\ 1/5	3
\ 1/15	1
Total:	4.

Therefore 1/5 and 1/15 is what is to be added to it [i.e., the given number].
Example of Proof (*tp n śyty*).[19]
Therefore 2/3 1/5 1/15 1/15 complete to (i.e., make) 1. For [when applied to 15 these fractions are equal to the numbers] 10, 3, 1, 1 [which make 15].

[Problem 22]
Complete 2/3 1/30 to 1.
[Applied to 30, the 2/3 1/30 equals] 21.
The Total of the excess of it [i.e., 30 over 21] is 9.
Multiply 30 in order to find 9.

1	30
\ 1/10	3
\ 1/5	6 *[cont.]*

Total: 9.

Therefore 1/5 1/10 is to be added to it [in order to make the completion].

Therefore the complete addition is of 2/3 1/5 1/10 1/30 to make 1, [for applying them to 30, these fractions are equal to] 20, 6, 3, and 1 [making 30].

[Problem 23]

Complete 1/4 1/8 1/10 1/30 1/45 to 2/3.

[Applied to 45, these are equal to] **11 1/4, 5 1/2 1/8, 4 1/2, 1 1/2, and 1** [which require 6 1/8 more to make up 2/3 of 45, i.e., 30, and 6 1/8 is equal to 1/9 1/40 of 45].

Therefore 1/9 1/40 is what is to be added to it to make 2/3 [of 1] since 1/4 1/8 1/9 1/10 1/30 1/40 1/45 and 1/3 make 1 [for applied to 45, these fractions are equal to] **11 1/4, 5 1/2 1/8, 5, 4 1/2, 1 1/2, 1 1/8, 1, and 15.**[20]

[PROBLEMS 24-29: QUANTITY (ʿḥʿ) PROBLEMS[21]]

[Problem 24]

A quantity with 1/7 of it added to it becomes (*ḫpr*) 19.[22] [What is the quantity?]

[Assume 7.]

\ 1	7
\ 1/7	1
[Total:	8.]

[As many times as 8 must be multiplied to give 19, so many times 7 must be multiplied to give the required number.]

1	8
\ 2	16
1/2	4
\ 1/4	2
\ 1/8	1

[Total 2 1/4 1/8.][23]

\ 1	2 1/4 1/8

\ 2	4 1/2 1/4
\ 4	9 1/2

Do it as follows: The quantity [is] 16 1/2 1/8 [and its] 1/7 [is] 2 1/4 1/8. [Hence this checks out since the] Total [is] 19 [as originally specified].

[Problem 25]

A quantity with 1/2 of it added to it becomes 16.[24] [What is the quantity?]

[Assume 2.]

\ 1	2
\ 1/2	1
Total:	3.

[As many times as 3 must be multiplied to give 16, so many times 2 must be multiplied to give the required number.]

\ 1	3
2	6
\ 4	12
2/3	2
\ 1/3	1

Total: 5 1/3.

1	5 1/3
\ 2	10 2/3

Do it as follows: **The quantity** [is] 10 2/3, [and its] 1/2 is 5 1/3. [Hence this checks out since the] Total [is] 16 [as originally specified].

[Problem 26]

A quantity with 1/4 of it added to it becomes 15.[25]

[Assume 4.] [That is] multiply 4, making 1/4, namely 1, [so that the] Total is 5 [proceeding in the usual manner:

\ 1	4
\ 1/4	1
Total:	5]. *[cont.]*

[As many times as 5 must be multiplied to make 15, so many times 4 must be multiplied to give the required number.]

Operate on 5 to find 15

\ 1	5
\ 2	10

Total: 3.

Multiply 3 times 4.

1	3
2	6
\ 4	12

This becomes 12. [And find its 1/4:]

1	12
\ 1/4	3
Total:	15.

[Hence] **the quantity is** 12 and its 1/4 is 3 and **the total is 15.** [This checks out since the sum agrees with what was originally specified.]

[Problem 27]

A quantity with 1/5 of it added to it becomes 21.[26] [What is the quantity?]

[Assume 5.]

\ 1	5
\ 1/5	1
Total:	6.

[As many times as 6 must be multiplied to give 21, so many times 5 must be multiplied to give the required number.]

\ 1	6
\ 2	12
\ 1/2	3

Total: 3 1/2.

\ 1	3 1/2
2	7
\ 4	14.

The quantity is 17 1/2 and 1/5 of it is 3 1/2 and **the total is 21.**

[This checks out since the sum agrees with what was originally specified.]

[Problem 28]

[If a quantity] and 2/3 of it are added together, and from the sum is subtracted 1/3 of the sum, 10 remains.[27] [What is the quantity?] Make 1/10 of 10; this becomes 1. Subtract 1 from 10 and the remainder is 9 [which is the desired quantity]. 2/3 of 9 is 6, which added to 9 makes 15. 1/3 of it (15) is 5, and 1/3 of 15 taken away from 15 leaves 10. Do it in the following way.[28]

[Problem 29]

[A quantity and its 2/3 are added together, and 1/3 of the sum is added; then 1/3 of this sum is taken and the result is 10. What is the quantity?][29]

\ 1	10
\ 1/4	2 1/2
\ 1/10	1
Total (i.e., quantity) [is]	**13 1/2.**
2/3	9
Total:	22 1/2.
1/3	7 1/2
Total:	30
2/3	20
1/3	10.

[PROBLEMS 30-34. DIVISIONS BY FRACTIONAL EXPRESSIONS[30]]

[Problem 30]

If the scribe (*sš*) **says to you** "What is the quantity of which 2/3 and 1/10 will make 10,"[31] let him hear [the following:]

Multiply 2/3 1/10 [by multipliers] in order to find [i.e., so as to get] 10.

\ 1	2/3 1/10
2	1 1/3 1/5 *[cont.]*

\4	3 1/15
\8	6 1/10 1/30

Total: 13 1/30 [still remains in order to produce 10].[32]

The making (*ỉrt*) of 1/30 times 1/23 for the finding of 2/3 1/10.

The total is the [desired] quantity so stated; [it is] 13 1/23.

1	13 1/23
\2/3	8 2/3 1/46 1/138
\1/10	1 1/5 1/10 1/230

Total: 10.

[Problem 31]

A quantity (*ʿḥʿ*), 2/3 of it, 1/2 of it, and 1/7 of it added together become 33.[33] [What is the quantity?]

[Multiply 1 2/3 1/2 1/7 so as to get 33.][34]

1	1 2/3 1/2 1/7
\2	4 1/3 1/4 1/28
\4	9 1/6 1/14
\8	18 1/3 1/7
1/2	1/2 1/3 1/4 1/14
\1/4	1/4 1/6 1/8 1/28

The total [of multipliers is 14 1/4 and that multiplied by 1 2/3 1/2 1/7] is 32 1/2 and a remainder of 1/2 [which is equal to the fractions 1/7, 1/8, 1/14, 1/28, and 1/28 plus an additional product to be determined].

1/7, 1/8, 1/14, 1/28, 1/28 [taken as parts of 42 are]

6, 5 1/4, 3, 1 1/2, 1 1/2, [which make in all] 17 1/4. [But we still require 3 + 1/2 + 1/4 more to make] 21, [which is] 1/2 [of 42].[35] [Then apply 1 2/3 1/2 1/7 to 42 as follows:]

\1	42
\2/3	28
\1/2	21
\1/7	6

Total: 99 [*should be* 97!].

[Since the total of the products of these multipliers applied to 42 is 97, hence 1/42 of 42, which is 1, will be 1/97 of the total. And hence 3 1/2 1/4 will be 3 1/2 1/4 times as much. Hence to find the fractional multipliers that will produce that final product, we need to find the missing fractional multipliers which will add up to 3 1/2 1/4. This is not presented as a straight multiplication but by listing the desired multipliers as applied to 42 in order to find the equivalent fractions of 97:]

\ 1/97	1/42 [or] 1 [as a part of 42]
\ 1/56 1/679 1/776	1/21 [or] 2 [as a part of 42]
\ 1/194	1/84 [or] 1/2 [as a part of 42]
\ 1/388	1/168 [or] 1/4 [as a part of 42]

[Hence, after arranging the fractions in decreasing order, the total unknown quantity sought is 14 1/4 1/56 1/97 1/194 1/388 1/679 1/776, which when multiplied by 1 2/3 1/2 1/7 makes the] Total 33 [as given in the enunciation of the problem].

[Problem 32]

A quantity, 1/3 of it, and 1/4 of it added together become 2.[36] [What is the quantity?]

[Multiply 1 1/3 1/4 so as to get 2.][37]

1	1 1/3 1/4	[as applied to 144 is]	228
\ 2/3	1 1/18	[ditto]	152
\ 1/3	1/2 1/36	[ditto]	76
\ 1/6	1/4 1/72	[ditto]	38
\ 1/12	1/8 1/144	[ditto]	19
\ 1/228	1/144	[ditto]	1
\ 1/114	1/72	[ditto]	2.

[Adding together the multipliers we find that the] total [of the required quantity is] 1 1/6 1/12 1/114 1/228.

[The number 144 used in the above table was determined as follows:][38]

1	12
2	24
\ 4	48
\ 8	96
Total:	144. *[cont.]*

[The following calculations result when we relate our fractions to 144]:

\ 1	144
\ 1/3	48
\ 1/4	36
Total	228.

[The first four numbers in the third column of the initial table which are opposite the checked fractions and which express the parts of 1 1/3 1/4 as applied to 144 add up to 285. But 3 more parts are needed to total 288, which is 2 times 144. Hence the last two items in that list of numbers that add to 3 are opposite the last two checked fractional multipliers completing the desired unknown quantity; these fractions being 1/228 and 1/114. Thus the complete unknown multiplier, the so-called "quantity" or *aha* which is the objective of the problem is that stated at the end of the first table: 1 1/6 1/12 1/114 1/228. To test that solution the author substitutes this value of the unknown in the enunciation of the problem to get 2.]

Example of proof (*tp n šyty*)

\ 1	1 1/6 1/12 1/114 1/228
[2/3	2/3 1/9 1/18 1/171 1/342]
\ 1/3	1/3 1/18 1/36 1/342 1/684
[1/2	1/2 1/12 1/24 1/228 1/456]
\ 1/4	1/4 1/24 1/48 1/456 1/912.

The total is 1 1/2 1/4 [and a series of smaller fractions. 1 1/2 1/4 taken from 2 leaves] a remainder of 1/4. [Apply the smaller fractions to 912. Now the smaller fractions are:]
1/12, 1/114, 1/228, 1/18, 1/36, 1/342, 1/684, 1/24, 1/48, 1/456, and 1/912; [as parts of 912, they are equal to:]
76, 8, 4, 50 2/3, 25 1/3, 2 2/3, 1 1/3, 38, 19, 2, and 1, the total [of which] is 228, i.e., 1/4 of 912. For

1	912
1/2	456
1/4	228.

[Problem 33]

A quantity, 2/3 of it, 1/2 of it, and 1/7 of it, added together become 37.[39] [What is the quantity?]
[Multiply 1 2/3 1/2 1/7 so as to get 37.][40]

1	1 2/3 1/2 1/7
2	4 1/3 1/4 1/28
4	9 1/6 1/14
8	18 1/3 1/7
\ 16	36 2/3 1/4 1/28 [and applying these three fractions to 42 we have:] **28, 10 1/2, 1 1/2.**

[1	42
\ 2/3	28
1/2	21
\ 1/4	10 1/2
\ 1/28	1 1/2.

The total [of the checked products] is 40; the remainder is 2, [or 1/21 of 42].
[Since 1 2/3 1/2 1/7 applied to 42 gives 97,[41] we note further that:]

1/97	1/42 [or] 1 [as a part of 42]
\ 1/56 1/679 1/776	1/ 21[or] 2 [as a part of 42].

[This last number, 1/21, with the product already obtained gives] a total of 37 [which is equal to the number specified in the enunciation of the problem. So the quantity sought is 16 1/56 1/679 1/776.]

Example of proof (with the values of the small fractions applied to 5432 expressed in bold-faced numbers below the fractions).

1	16	1/56	1/679	1/776		
		97	**8**	**7**		
2/3	10 2/3	1/84	1/1358	1/4074[42]	1/1164	
		64 2/3	**4**	**1 1/3**	**4 2/3**	
1/2	8	1/112	1/1358	1/1552		
		48 1/2	**4**	**3 1/2**		
1/7	2 1/4 1/28		1/392		1/4753	1/5432
			12 (!, 13) 1/2 1/4 1/14 1/28		**1 1/7**	**1.**

[The whole numbers and larger fractions add up to:] *[cont.]*

36	2/3	1/4	1/28 and the remainder to 1/28	1/84.
	3621 1/3	**1358**	**194**	**194 64 2/3**

[with the fractions applied to 5432 again given in boldface numbers]. [For regarding the larger fractions:]

1	5432
2/3	3621 1/3
1/2	2716
1/4	1358
1/28	194

Total: 5173 1/3. And [that of] the remainder is 258 2/3.

[Problem 34]

A quantity, 1/2 of it, and 1/4 of it, added together, become 10.[43] [What is the quantity?]

[Multiply 1 1/2 1/4 so as to get 10.]

\ 1	1 1/2 1/4
2	3 1/2
\ 4	7
\ 1/7	1/4
1/4 1/28	1/2
\ 1/2 1/4	1.

The total is the required quantity: 5 1/2 1/7 1/14.

Example of proof:

\ 1	5 1/2 1/7 1/14
\ 1/2	2 1/2 1/4 1/14 1/28
\ 1/4	1 1/4 1/8 1/28 1/56.

The total [of whole numbers and simpler fractions (powers of 1/2)] is 9 1/2 1/8; the remainder is 1/4 1/8. [The rest of the fractions follow, and their applications to 56 are given in boldface below them:]

1/7	1/14	1/14	1/28	1/28	1/56
8	**4**	**4**	**2**	**2**	**1**

and the total [of these last numbers is 21]. [Furthermore] 1/4 [applied to 56] is 14 and 1/8 is 7. [Their total is also 21. Hence the result obtained is correct.]

[PROBLEMS 35-38: DIVISION OF A HEQAT[44]]

[Problem 35]

I go down three times into the heqat-measure (*ḥḳ3t*), 1/3 of me is added to me, I return filled [i.e., having filled the heqat-measure]. Then what says this?[45]

The Procedure is as follows: [Assume 1. Multiplying by 3 1/3 we have]:

\ 1	1
\ 2	2
\ 1/3	1/3
Total:	3 1/3.

Call 1 out of 3 1/3.

1	3 1/3
\ 1/10	1/3
\ 1/5	2/3
Total:	**1.**

[The answer is 1/5 1/10.]

Example of proof:

\ 1	1/5 1/10
\ 2	1/2 1/10
\ 1/3	1/10
Total:	**1.**

[Expressing the result of the fractional parts 1/10 and 1/5 in *r* (i.e., 𓂋, "the part"), sometimes transliterated *re* or *ro*, of which the value is 1/320 part of a heqat. Like Chace, I use "ro" everywhere.]

1	320
\ 1/10	32
\ 1/5	64
Total:	96.

Example of proof:

\ 1	96
\ 2	192
\ 1/3	32 *[cont.]*

Total: 320.

[As expressed in Horus-eye fractions of a heqat (given here in Italics), namely, the fractions whose denominators were the powers of 2 down to 1/64 and which were expressed by special notations,[46] the result makes of grain *1/4 1/32 1/64* heqat and 1 ro.]

[Proof in the] making of grain [i.e., in corn-measure using the Horus-eye form:]

\ 1	*1/4 1/32 1/64* [heqat and] 1 [ro]
\ 2	*1/2 1/16 1/32* [heqat and] 2 [ro]
\ 1/3	*1/16 1/32* [heqat and] 2 [ro]
Total:	a heqat [i.e., 1 heqat].

[Problem 36]

I go down three times [into the heqat-measure[47]]; 1/3 of me and 1/5 of me are added to me, and I return having filled the measure.[48] What is the quantity (*ʿḥʿ*) that says this?

[Assume 1. Multiplying by 3 1/3 1/5 we have]

1	1
1	1
1	1
1/3	1/3
1/5	1/5
Total:	3 1/3 1/5.

[Get 1 by operating on 3 1/3 1/5. Apply this to 30; it makes 106. Multiply 106 so as to get 30.]

1	106
1/2	53
\ 1/4	26 1/2
\ 1/106	1
\ 1/53	2
\ 1/212	1/2
Total:	[30, i.e., the whole of 30, or] 1.

[The answer is 1/4 1/53 1/106 1/212.]

[Proof:]

1	1/4 1/53 1/106 1/212

2	1/2 1/30 1/318 1/795 1/53 1/106
1/3	1/12 1/159 1/318 1/636
1/5	1/20 1/265 1/530 1/1060.

[The larger fractions are 1/2 and 1/4. In order to get 1 we should have for the sum of the remaining fractions 1/4. To get this we apply the smaller fractions to 1060, and beneath these fractions they are written as parts of 1060 in red in the papyrus and thus here in boldface type:]

1/53	1/106	1/212			
20	**10**	**5**			[or in total] **35**
1/30	1/318	1/795	1/53	1/106	
35 1/3	**3 1/3**	**1 1/3**	**20**	**10**	[or in total] **70**
1/12	1/159	1/318	1/636		
88 1/3	**6 2/3**	**3 1/3**	**1 2/3**		[or in total] **100**
1/20	1/265	1/530	1/1060		
53	**4**	**2**	**1**		[or in total] **60**[49]
			[Grand total of parts:]		**265**[50]

[which is] 1/4 [of 1060, for]

1	1060
1/2	530
1/4	265
1/4	265
Total:	1060.

[Problem 37][51]

I go down three times into the heqat measure, 1/3 of me is added to me, 1/3 of 1/3 of me is added to me, and 1/9 of me is added to me; I return having filled the heqat-measure. **Then what is it that says this? You shall hear [the answer].**

[Assume 1. Multiplying by the given expression we have]

1	1
2	2
1/3	1/3
1/3 of 1/3	1/9
1/9	1/9 *[cont.]*

Total:	**3 1/2 1/18.**

Call 1 out of 3 1/2 1/18.

1	3 1/2 1/18
1/2	1 1/2 1/4 1/36
\ 1/4	1/2 1/4 1/8 1/72
1/8	1/4 1/8 1/16 1/144
1/16	1/8 1/16 1/32 1/288
\ 1/32	1/16 1/32 1/64 1/576
Total:	1 [for if we] add

1/2 1/4 1/8 1/72 1/16 1/32 1/64 1/576 [we get 1. Now]

8 36 18 9 1

[are the values of the smaller fractions under which they are written when taken as parts of 576. These parts] total 72 [which is] 1/8 [of 576. Therefore the answer is 1/4 1/32.]

Example of Proof:

1	1/4 1/32
2	1/2 1/16
1/3	1/12 1/96
1/3 of 1/3	1/36 1/288
1/9	1/36 1/288.
Total:	1 [for if we] **add**

1/2 1/4 1/32 1/16 1/12 1/96 1/36 1/288 1/36 1/288 [they equal 1. Now]

9 18 24 3 8 1 8 1

[are the values of those smaller fractions below which they are written when they are taken as parts of 288. These parts] total 72 [which is] 1/4 [of 288].

[We can, as in Problem 35, express the result in ro:]

Total:

[1]	320
1/2	160
\ 1/4	80
1/8	40
1/16	20
\ 1/32	10

Total:	90.

Example of proof:

\ 1	90
\ 2	180
\ 1/3	30
\ 1/3 of 1/3	10
\ 1/9	10
Total:	320.

[It amounts in grain to *1/4 1/32* heqat.]
[The proof] produced in the grain-measure or Horus-eye fractions [noted in Italics:]

\ 1	*1/4 1/32*
\ 2	*1/2 1/16*
\ 1/3	*1/16 1/32*
\ 1/3 of 1/3	*1/32*
\ 1/9	*1/32*
Total:	*1/2 1/8 1/4 1/8.*

[Problem 38][52]

I have gone down three times into the heqat-measure, 1/7 of me **is added to me. I return having filled the heqat-measure.** [What is it that says this?]

[Assume 1. Multiplying by the given expression we have]

\ 1	1
\ 2	2
\ 1/7	1/7
Total:	**3 1/7**.

Call 1 out of 3 1/7.

1	3 1/7
1/22	1/7 [for] 1/7 times 22 is 3 1/7
1/11	1/4 1/28
1/6 1/66	1/2 1/14
Total:	**1**.

[Therefore the answer is 1/6 1/11 1/22 1/66.]
Example of proof: *[cont.]*

1	1/6 1/11 1/22 1/66
2	1/2 1/11 1/33 1/66
1/7	1/22 [for] 1/22 of 7 is the expression we have obtained.

[Expressing the result in ro:]

1	320
2/3	213 1/3
1/3	106 2/3
\ 1/6	53 1/3
\ 1/11	29 1/11
\ 1/22	14 1/2 1/22
\ 1/66	4 2/3 1/6 1/66
Total:	101 2/3 1/11 1/22 1/66.

Example of proof [in ro]:

\ 1	101 2/3 1/11 1/22 1/66
\ 2	203 1/2 1/11 1/33 1/66
\ 1/7	14 1/2 1/22
Total:	1 [heqat, i.e., 320 ro]

Making of grain (*ir n šš*) [i.e., its amount in grain, using the Horus-eye fractions in Italics and ros in Roman type]

1	*1/4 1/16* [heqat]	1 2/3 1/11 1/22 1/66 [ro]
2	*1/2 1/8* [heqat]	3 1/2 1/11 1/33 1/66 [ro]
1/7	*1/32* [heqat]	4 1/2 1/22 [ro].

Total: 319 2/3 [ros for the larger portions; the smaller fractions are] 1/11, 1/11, 1/22, 1/22, 1/33, 1/66, and 1/66. [They total] 1/3, [for if we express them as parts of 66, they are successively:] **6, 6, 3, 3, 2, 1, and 1** [which total] **22** [and 22 is 1/3 of 66; and so the parts in total are 319 2/3 plus 1/3, which make 320 ros, or 1 heqat].

[PROBLEMS 39-40: DIVISION OF LOAVES AND ARITHMETICAL PROGRESSION]

[Problem 39]

Example of making excess (*tp n lrt twnw*) [or, of making the difference of share] when 100 loaves are for 10 men, 50 for 6, and 50 for 4. What is the excess of shares ?[53]

[Multiply 4 so as to get 50.]

1	4
\ 10	40
\ 2	8
\ 1/2	2

Total: 12 1/2.

[Multiply 6 so as to get 50.]

1	6
2	12
4	24
\ 8	48
\ 1/3	2

Total: 8 1/3.

[Therefore each of the men in the group of 4 gets] 12 1/2 loaves [and each of the men in the group of 6 gets] 8 1/3 [and the first amount is listed 4 times and the second amount is listed 6 times]. The excess [of the share of each man in the group of 4 over the share of each man in the group of 6] is 4 1/6.[54]

[Problem 40]

[Divide] **100 loaves** among 5 men [in such a way that the shares received will be in arithmetical progression and that] 1/7 of [the sum of] the largest three shares is [equal to the sum of] the smallest two. What is the [common] excess [or difference of the shares]?

The procedure is as follows, [if we assume first that] the excess [or difference] is 5 1/2.[55] [Then the amounts that the five men receive are]

23, 17 1/2, 12, 6 1/2, 1; total 60.

[As many times as is necessary to multiply 60 to make 100, so many times must the terms noted above be multiplied to find the correct terms of the series.] *[cont.]*

\1	60
\2/3	40
Total: 1 2/3	100.

[Then] multiply [the above assumed terms] by 1 2/3 [as follows:]

23	it becomes	38 1/3
17 1/2	it becomes	29 1/6
12	it becomes	20
6 1/2	it becomes	10 2/3 1/6
1	it becomes	1 2/3
Total: 60	it becomes	100.

[And so the common excess or difference between any two terms is 9 1/6.]

[PROBLEMS 41-46: PROBLEMS OF VOLUME]

[Problem 41]

Example of making (i.e., calculating the volume of a) round (i.e., cylindrical) granary of [diameter] 9 and [height] 10.[56]

Take away 1/9 of 9, namely, 1; the remainder is 8. Multiply 8 times 8; it makes 64. Multiply 64 times 10; it makes 640 [cubic] cubits. Add 1/2 of it to it; it makes 960: **the calculation of [the content of] it in khar** (*ḫ3rw*). Take 1/20 of 960, namely, 48. This is what goes into it in [the number of hundreds of] quadruple-heqats, (*4-ḥḳ3t*), [i.e.,] **in grains, 4800 heqats.**

Method of reckoning it (*ḳy n šśmtf*).

1	8
2	16
4	32
\8	64.

1	64
\10	640
\1/2	320

Total:	960
\ 1/10	96
\ 1/20	48.

[Problem 42]

[Find the volume of] a round [i.e., cylindrical] granary of [base-diameter] 10 and [height] 10.

Take away 1/9 of 10, i.e., 1 1/9; the remainder is 8 2/3 1/6 1/18. Multiply 8 2/3 1/6 1/18 times 8 2/3 1/6 1/18; it makes 79 1/108 1/324. Multiply 79 1/108 1/324 times 10; it makes 790 1/18 1/27 1/54 1/81 [cubic cubits]. Add 1/2 of it to it; it makes 1185 1/6 1/54, [This is its content or volume in khar.] [We find that] 1/20 of it is 59 1/4 1/108. [Multiplying this times 100 heqat,] we find what goes into this in quadruple heqat is, namely, **5925 heqat of grains.**[57]

Method of the reckoning of it:

1	8 2/3 1/6 1/18
2	17 2/3 1/9
4	35 1/2 1/18
\ 8	71 1/9
\ 2/3	5 2/3 1/6 1/18 1/27
1/3	2 2/3 1/6 1/12 1/36 1/54
\ 1/6	1 1/3 1/12 1/24 1/72 1/108
\ 1/18	1/3 1/9 1/27 1/108 1/324
Total:	79 1/108 1/324.

1	79 1/108 1/324
10	790 1/18 1/27 1/54 1/81
1/2	395 1/36 1/54 1/108 1/162
Total:	1185 1/6 1/54
1/10	118 1/2 1/54
1/20	59 1/4 1/108.

[Problem 43, as given in the Rhind Papyrus][58] *[cont.]*

A round (i.e., cylindrical) granary of 9 cubits in its height (! diameter?) and 6 in its breadth (! height?); what is the content [or volume] of grain that goes into it?

The procedure is as follows:

Take away 1/9 from 9; the remainder is 8. Add to 8 its 1/3; it makes 10 2/3. Multiply 10 2/3 times 10 2/3; it makes 113 2/3 1/9. Multiply 113 2/3 1/9 times 4, 4 being 2/3 of 6 cubits which is its breadth (! height?). 455 1/9 **is the amount [of the volume] in khar.** Find 1/20 of the amount of it in khar; this is the amount that goes into it of quadruple heqat, i.e., **grain to the amount of 2200 heqat [and] 50, 25**[59]**, 1/2, 1/32, 1/64 [heqat, and] 2 1/2,1/4, 1/36 ro.**

Method of reckoning it:

\ 1	8
2/3	5 1/3
\ 1/3	2 2/3
Total:	10 2/3.

1	10 2/3
\ 10	106 2/3
\ 2/3	7 1/9
Total:	113 2/3 1/9.

1	113 2/3 1/9
2	227 1/2 1/18
\ 4	455 1/9.

1	455 1/9
1/10	45 1/2 1/90
\ 1/20	22 1/2 1/4 1/45 (!, 1/180).

[Problem 43—A reconstruction based on earlier discussions by Griffith, Schack-Schackenberg, and Peet][60]

A round (i.e., cylindrical) granary with a diameter of 9 cubits and a height of 6; what is the amount in grain that goes into it?

The procedure is as follows:

Add to the diameter 1/3 of it; it makes 12. Multiply 12 times 12; this makes 144. Multiply 144 times 4, 4 being 2/3 of 6 cubits which is the height. This makes the volume in khar, 576.

Method of reckoning it:

\1	9
2/3	6
\1/3	3
Total:	12.

1	12
\2	24
\10	120
Total:	144.

1	144
2	288
\4	576.

[Problem 43—A better Reconstruction following Gillings][61]

A round (i.e., cylindrical) granary with a diameter of 8 cubits and a height of 6; what is the amount of grain that goes into it?

The procedure is as follows:

Add to the diameter its 1/3; it makes 10 2/3.......[From this point on the text presented in the Rhind Papyrus can be followed, i.e., the first of the three versions given here.]

[Problem 44]

Example of reckoning [the volume of] a rectangular granary, its length being 10, its breadth 10, and its height 10. What is the amount of grain that goes into it?

Multiply 10 times 10; it makes 100. Multiply 100 times 10; it makes 1000. Take 1/2 of 1000, namely 500, [and add it to 1000;] it makes 1500, its contents in khar. Take 1/20 of 1500; it makes 75, its contents in quadruple heqat, namely, **7500 heqat of grain.**

Example of the working out *[a continuation from the previous page]*:

1	10
10	100
1	100
10	1000
1	1000
1/2	500
1	1500
1/10	150
1/20	75.

[Proof:]

1	75
10	750
\ 20	1500
1/10 [of 1500]	150
1/10 of 1/10	15
2/3 of 1/10 of 1/10	10.

[Problem 45]

A [rectangular] granary into which have gone 7500 quadruple heqat of grain. What are its dimensions (*lit.*, by how much of it)?[62]

Multiply 75 times 20; it makes 1500. Take 1/10 of 1500, namely, 150; 1/10 of its 1/10, 15; 2/3 of 1/10 of its 1/10, 10. Therefore [the dimensions] of it (i.e., the granary) are 10 by 10 by 10.

[The procedure:]

1	75
10	750
20	1500 [which is its contents in khar]:
1	1500

1/10	150
1/10 of 1/10	15
2/3 of 1/10 of 1/10	10.[63]

[Problem 46]

A [rectangular] granary into which there have gone 2500 quadruple heqat of grain. What are its dimensions (*rḫt*, i.e., "amount")?

Multiply 25 times 20; it makes 500, the content of this [in khar]. Take 1/10 of it (i.e., of 500), namely, 50; its 1/20, 25; 1/10 of its 1/10, 5; 2/3 of 1/10 of its 1/10, 3 1/3. [Therefore, the dimensions] of the granary are 10 by 10 by 3 1/3.

The calculation of it:

1	25
10	250
20	500 [its contents in khar]

1	500
1/10	50
1/10 of 1/10	5
2/3 of 1/10 of 1/10	3 1/3.

[Therefore, the dimensions of] the granary, in cubits, are 10 by 10 by 3 1/3, as here [noted].

[DIVISION OF 100 HEQAT]

[Problem 47]

If the scribe says to you, "Let me know what is the result when 100 quadruple heqat of grain are divided by 10 [and its multiples] in a [rectangular or] circular granary." [Table ends on next page.]

1/10 becomes of grain	10 quadruple	heqat	
1/20 becomes of grain	5 quadruple	heqat	
1/30 becomes of grain	3 *1/4 1/16 1/64*	heqat	1 2/3 ro
1/40 becomes of grain	2 *1/2*	heqat	
1/50 becomes of grain	2	heqat	
1/60 becomes of grain	1 *1/2 1/8 1/32*	heqat	3 1/3 ro

1/70 becomes of grain 1 *1/4 1/8 1/32 1/64* heqat 2 1/14 1/21
[the line above is cont. from previous page]
1/42 ro
1/80 becomes of grain 1 *1/4* heqat
1/90 becomes of grain 1 *1/16 1/32 1/64* heqat 1/2 1/18 ro
1/100 becomes of grain 1 heqat.

[PROBLEMS 48-55: PROBLEMS OF AREA]

[Problem 48; see Fig. IV.2, Pl. 70]
[Compare the area of a circle (or, better, an octagon?[64]) and its circumscribing square.]

[Cir. of diam. 9 (or oct. = to sq. of side 8?)]		[Sq. of side 9]	
1	8 setjat[65]	\ 1	9 setjat
2	16 setjat	2	18 setjat
4	32 setjat	4	36 setjat
\ 8	64 setjat	\ 8	72 setjat
		Total:	81 setjat

[Problem 49; see Fig. IV.2, Pl. 71]
Example of reckoning area. If it is said to you: "What is the area of a rectangle of land of 10 khet by 2 (!, *should be* 1) khet?"
Proceed as follows:

1	1000 [cubits, i.e., 10 khet]
10	10,000
100	100,000
1/10	10,000
1/10 of 1/10	1000 [cubit-strips, i.e., each strip 1 cubit wide and 1 khet long.] This is its area.[66]

[Problem 50]
Example of producing [the area of] a round field of diameter of 9 khet. What is the reckoning (*lit., rḫt,* knowledge) of its area (*3ḥt*)?

Take away 1/9 of it (the diameter), namely 1; the remainder is 8. Multiply 8 times 8; it makes 64. [Therefore,] the amount of it in area is 64 setjat.

The procedure is as follows:

1	9
1/9	1;

this taken away leaves 8.

1	8
2	16
4	32
\8	64.

The amount of it in area is 64 setjat.[67]

[Problem 51; see Fig. IV.2, Pl. 73]

Example of producing (i.e., calculating) [the area of] a triangle *(spdt)* of land. If it is said to you: "What is the area of a triangle of 10 khet on the *mryt* (most likely, the "height" or "kathete"; less likely, the "side"?)[68] of it and 4 khet on the base of it?"

The Procedure is as follows:

1	400 [cubits, i.e., 4 khet]
1/2	200 [cubits, i.e., 2 khet]

1	1000 [cubits, i.e., 10 khet]
2	2000.

Its area is 20 setjat.[69]

Take 1/2 of 4, namely, 2, in order to get [one side of] its [equivalent] rectangle. Multiply 10 [the other side of the rectangle] times 2; this is its area [i.e., the area of the rectangle and thus of the triangle].

[Problem 52; see Fig. IV.2, Pl. 74]

Example of a truncated triangle (i.e., a trapezoid). If it is said to you: "What is the area of a truncated triangle of land of 20 khet in its height [or, side?], 6 khet in its base, 4 khet in its truncating line?"

Add its base to its truncating line; it makes 10. Take 1/2 of 10, i.e., 5, in order to get [one side of] its [equivalent] rectangle. Multiply 20 [the other side of its rectangle] times 5; it makes 10 (10 ten-setjat). This is the area.

The procedure is as follows:

1	1000 [cubits, i.e., 10 khet]
1/2	500
\ 1	2000
2	4000
\ 4	8000
Total:	10,000 [cubit-strips, as in the preceding problem].

Its area is 100 setjat (10 ten-setjat[70]).

[Problem 53; see Fig. IV.2LL, Pl. 75]

[Areas of sections of a compound trapezoidal-triangular figure?][71]

\ 1	4 *1/2*[72]	setjat
\ 2	9	setjat
1/2	2 *1/4*	setjat
\ 1/4	1 *1/8*	setjat
Total:	5 *1/2 1/8* [! 14 *1/2 1/8*]	setjat.

1/10 of it is 1 *1/4 1/8* setjat [and]10 cubit-strips.[73]

Take away the 1/10 of it. This [i.e., the remainder] is the area of it [i.e., of the middle trapezoidal figure].

[Turning to the triangular section: i.e., to the triangle with altitude of 7 and base of 2 1/4; it is calculated as follows:[74]]

1	7	setjat
\ 2	14	setjat
1/2	3 *1/2*	setjat
\ 1/4	1 *1/2 1/4*	setjat
Total:	15 *1/2 1/4*	setjat.
1/2	7 *1/2 1/4 1/8*	setjat [and this is the area of the triangular element].

[The calculation of the third section, be it trapezoidal or rectangular, is missing.]

[Chace's reconstruction, with some alterations, of the first set of calculations of Problem 53 in order to approximate the area of the first trapezoidal sections][75]

[Areas of sections of an isosceles triangle.]

\ 1	4 *1/2*[76]	setjat
\ 2	9	setjat
\ 1/2	2 *1/4*	setjat
Total:	15 *1/2 1/4*	setjat
1/10	1 *1/2*	setjat [and] 7 1/2 cubit-strips.

Its 1/10 taken away leaves the area: [14 *1/8* [[and]] 5 cubit-strips.]

[Following along with Chace's reconstruction, we would note that the next trapezoidal section, i.e., the middle section in Fig. IV.5c, is not presented in the papyrus, since he had taken the first set of calculations to be concerned with the determination of the area of the first trapezoidal section. But obviously if the first trapezoidal section was known along with the triangular section, then they could be subtracted from the whole isosceles triangle (which could easily be found as was the triangular section). Hence the remainder would be the area of the middle trapezoidal figure.[77] The setjat-fractions in Italics in Problems 53-55 are normal hieratic signs rather than Horus-eye signs of like value, namely *1/2 1/4 1/8*.]

[Problem 54]

The subtraction (*ḫbt*) of the area [7 setjat] from 10 fields [i.e., What equal areas should be taken from 10 fields if the sum of these areas is to be 7 setjat?].[78]

[Multiply 10 so as to get 7.]

1	10
\ 1/2	5
\ 1/5	2

[Total: 1/2 1/5.]

[The above total equals *1/2 1/8* setjat plus 7 1/2 cubit-strips.]

[Proof:]

1	*1/2 1/8*	setjat	7 1/2 cubit-strips (*mḥ*)
\2	1 *1/4 1/8*	setjat	2 1/2 cubit-strips
4	2 *1/2 1/4*	setjat	5 cubit-strips
\8	5 *1/2*	setjat	10 cubit-strips
Total:	7	setjat.	

[Problem 55]

The subtraction of the area 3 setjat from 5[79] [i.e., What equal areas should be taken from 5 fields if the sum of these areas is to be 3 setjat?].[80]

Multiply 5 so as to get 3.[81]

1	5
1/2	2 1/2
1/10	1/2

Total: 1/2 1/10.

[Expressed as parts of a setjat and cubit-strips this is *1/2* setjat 10 cubit strips.]

[Proof:]

\1	*1/2*	setjat	10 cubit-strips
2	1 *1/8*	setjat	7 1/2 cubit-strips
\4	2 *1/4 1/8*	setjat	2 1/2 cubit-strips
	Total:	3 setjat. Thus the area is 3 setjat.	

[PROBLEMS 56-60. PYRAMIDS; THE RELATION OF THE LENGTHS OF TWO SIDES OF A TRIANGLE][82]

[Problem 56]

Example of reckoning a pyramid (*mr*) whose base-side (*wḫ3-ṯbt*) is 360 [cubits] and whose altitude (*pr-m-wś*) is 250 [cubits]. Cause that I know (i.e., calculate) its seqed (*sḳd*, also transcribed as seked), (i.e., slope[83]). [See Fig. IV.2mm, Plate 78.][84]

Take 1/2 of 360 and the result is 180. Multiply 250 so as to find 180. It makes 1/2 1/5 1/50 of a cubit. A cubit is 7 palms. Multiply 7 as follows:

1	7
1/2	3 1/2
1/5	1 1/3 1/15
1/50	1/10 1/25.

The seqed is 5 1/25 palms.

[Problem 57]

[In] **a pyramid** whose base-side is 140 [cubits] and whose seqed is 5 palms 1 finger [per unit of height], what is its altitude? [See Fig. IV.2nn, Plate 79.]

Divide 1 cubit by the seqed doubled, which is 10 1/2. Multiply 10 1/2 so as to get 7, for note that the latter is 1 cubit; operating on 10 1/2 we find that 2/3 of 10 1/2 is 7. Operating on 140, which is the base-side, we find that 2/3 of 140 is 93 1/3, and behold this is its altitude.[85]

[Problem 58]

In **a pyramid** whose altitude is 93 1/3, make known the seqed of it when its base-side is 140 [cubits]. [See Fig. IV.2nn, Plate 80.]

Take 1/2 of 140, which is 70. Multiply 93 1/3 so as to get 70. 1/2 of 93 1/3 is 46 2/3. 1/4 of it is 23 1/3. Take 1/2 1/4 of a cubit. Operate on 7: 1/2 of it is 3 1/2; 1/4 of it is 1 1/2 1/4; the total is 5 palms 1 finger. This is the seqed.

Working out:

1	93 1/3
\ 1/2	46 2/3
\ 1/4	23 1/3

Total 1/2 1/4.

Produce 1/2 1/4 of a cubit, the cubit being 7 palms.

1	7
1/2	3 1/2
1/4	1 [1/2] 1/4
Total:	5 palms 1 finger, which is the seqed.

[Problem 59] [cont.]

[In] **a pyramid** whose base-side is 12 [cubits] and whose altitude is 8 [cubits] what is its seqed? [See Fig. IV.2oo, Plate 81.]
Multiply 8 so as to get 6, which is 1/2 the base-side.[86]

1	8
\ 1/2	4
\ 1/4	2

Total 1/2 1/4.
Take 1/2 1/4 of 7; this is a cubit.

1	7
\ 1/2	3 1/2
\ 1/4	1 1/2 1/4.

The result is 5 palms 1 finger, which is its seqed.

[Problem 59B]

If you construct a pyramid with base-side 12 [cubits] and with a seqed of 5 palms 1 finger, what is its altitude?[87]
Operate on the double of 5 [palms] 1 [finger], which is 10 1/2, so as to get 1 cubit; a cubit is 7 palms. 2/3 of 10 1/2 is 7. Operating on 12, [we find that] 2/3 of it is 8, and this is the altitude.

[Problem 60]

[In] **a pillar** (*iwn*) [*or perhaps* a cone?][88] with a base-side (*sntt*) [*or perhaps* a diameter?] of 15 cubits and a height of 30 [cubits], what is its seqed? [See Fig. IV.2oo, Plate 82.[89]]
Take 1/2 of 15; it is 7 1/2. Operate on 30 so as to get 7 1/2. The result is 1/4, which is the seqed.[90]

[PROBLEMS 61-84: MISCELLANEOUS ARITHMETICAL DETERMINATIONS]

[Problem 61]

[Table for the multiplication of fractions][91]

2/3 of 2/3 is 1/3 1/9
1/3 of 2/3 is 1/6 1/18
2/3 of 1/3 is 1/6 1/18

2/3 of 1/6 is 1/12 1/36
2/3 of 1/2 is 1/3
1/3 of 1/2 is 1/6
1/6 of 1/2 is 1/12
1/12 of 1/2 is 1/24
1/9 of 2/3 is 1/18 1/54; 1/9, 2/3 of it (or 2/3 of 1/9) is 1/18 1/[54]
....
1/[5], 1/4 of it is 1/20
1/7, 2/3 of it is 1/14 1/42
1/7, 1/2 of it is 1/14
1/11, 2/3 of it is 1/22 1/66, 1/3 of it is 1/33
1/11, 1/2 of it is 1/22, 1/4 of it is 1/44.

[Problem 61B]

[Rule for] taking 2/3 of an uneven fraction (i.e., the reciprocal of an odd number). If it is said to you "What is 2/3 of 1/5?" you take the reciprocals of 2 times 5 and 6 times 5. You do the same thing to get 2/3 of the reciprocal of any odd number.

[Problem 62]

Example of reckoning a bag containing various precious metals. If it is said to you: "A bag containing [equal weights] of gold, silver, and lead is bought for 84 shaty (*š'ty*), what is the amount of each precious metal." As for what is given for a deben of gold, it is 12 shaty, for silver it is 6 shaty, and for lead it is 3 shaty.

Add together that which is given for a deben[92] of each precious metal and the result is 21. Multiply 21 so as to get the 84 shaty for which the bag was bought. The result is 4, which is the number of debens of each precious metal.[93]

[Check out the answer as follows:]

Multiply 12 by 4; the result is 48 [shaty] for the gold in it (i.e., the bag).

[Multiply] 6 [by] 4; [the result is] 24 [shaty] for the silver [in the bag].

[Multiply] 3 [by] 4; [the result is] 12 [shaty] for the lead [in the bag.]

[Multiply] 21 [by] 4; the total [price is] 84 [shaty, as was noted above].

[Problem 63]

[Example of dividing] 700 loaves among 4 men, [with] 2/3 to 1 [man], 1/2 to another, [1/3 to the third, and 1/4 to the 4th]. Let me know the share of it for each [man].

Add 2/3, [1/2,] 1/3, and 1/4; it makes 1 1/2 1/4. Call up 1 out of 1 1/2 1/4 (i.e., get 1 by operating on 1 1/2 1/4); the result is 1/2 1/14. Take 1/2 1/14 of 700, namely 400. Take 2/3 of 400, namely 266 2/3; [then] 1/2 of 400, namely 200, [then] 1/3 of 400, namely 133 1/3, [and finally] 1/4 of 400, namely 100. [Hence you now have] the share of each [one of the men]

The procedure is as follows:

The quantity is 700.

1/2 1/14 [of it]	400
2/3 of 400 for 1	266 2/3
1/2 of 400 for 1	200
1/3 of 400 for 1	133 (*corr. ex* 113) 1/3
1/4 of 400 for 1	100
Total:	700.

[Problem 64]

Example of dividing (i.e., distributing) **excess** (i.e., difference) [*or* Example of determining an arithmetical progression].[94] If it is said to you: "There is 10 heqat of barley [to be divided] among 10 men in such a way that the excess of barley of each successive man over his predecessor is *1/8* heqat [i.e., there shall be an arithmetical progression with a common difference of 1/8 heqat]," [What is the share of each man?] .

The average share is 1 (*corr. ex 1/2)* heqat. Take 1 from 10, and the remainder is 9 [as the number of differences, i.e., 1 less than the number of men]. Take 1/2 of the [common] difference, namely

1/16 [heqat]. Multiply this by 9 and the result is *1/2 1/16* [heqat]. Add [this] to the average share [and this becomes 1 *1/2 1/16*, which is the largest share]. Subtract *1/8* heqat for each man until you reach the last one.

The procedure is as follows:

1 *1/2 1/16*, 1 *1/4 1/8 1/16*, 1 *1/4 1/16*, 1 *1/8 1/16*, 1 *1/16*, *1/2 1/4 1/8 1/16*, *1/2 1/4 1/16*, *1/2 1/8 1/16*, *1/2 1/16*, *1/4 1/8 1/16*.

The total is 10 heqat.

[Problem 65]

Example of dividing (*lit.*, making) 100 loaves among 10 men, [three of them:] a boatman, a foreman, and a door-keeper, each having a double [share]. [What is the share of each?]

The working-out of it: Add to the people [supplied] three [because of the double shares of three of them]; the result is 13. Multiply 13 so as to get 100 loaves; the result is 7 2/3 1/39. You say "This is the ration (*lit.*, eating) for 7 men, and the boatman, foreman, and door-keeper receive double [such a portion]."[95]

[The proof of this is in the following addition:] [7 2/3 1/39 taken seven times for the first 7 men and]

7 2/3 1/39 + 7 2/3 1/39, making 15 1/3 1/26 1/78 for [each of] the boatman, the foreman, and the door-keeper. The total is 100.

[Problem 66]

10 heqat of fat is given out for a year. What is the amount for one day thereof (i.e., of the year)?

Reduce the 10 heqat of fat into ro; the result is 3200. Reduce the year (*rnpt*) to days (*hrw*); the result is 365.[96] Get 3200 by operating on 365; the result is 8 2/3 1/10 1/2190, making *1/64* [heqat] 3 2/3 1/10 1/2190 ro as a share for a day.

The procedure is as follows[97]:

1	365
2	730
4	1460 *[cont.]*

[\ 8	2920]
\ 2/3	243 1/3
\ 1/10	36 1/2
\ 1/2190	1/6

Total: 8 2/3 1/10 1/2190.
You shall proceed in this way in any example like this.[90]

[Problem 67]

Example of reckoning (*ḥśb*) [with] tribute-[cattle] of a herdsman.[99]

Now this herdsman came to the cattle-numbering with 70 [tribute] cattle. The accountant said to this herdsman, "Indeed, this is a small herd of [tribute-]cattle which you have brought. Where is the large number of cattle that you owe?" The herdsman answered him, "What I have brought is 2/3 of 1/3 of the cattle you have committed to me. Count them for me and you will find that I have brought the full number of them."

The procedure is as follows:

1	1
2/3	2/3
1/3	1/3
2/3 of 1/3 is	1/6 1/18.

Get 1 by operating on 1/6 1/18.

1	1/6 1/18
2	1/3 1/9
\ 4	2/3 1/6 1/18
\ 1/2	1/9
Total: [4 1/2].	1

Multiply 70 by 4 1/2; it makes 315. These are the cattle committed to him.

[Proof:]

1	315
2/3	210
1/3	105
2/3 of 1/3	70.

These are the ones he brought.

[Problem 68]

If a scribe says to you: "4 foremen have received their grain [in the amount of] 100 great quadruple-heqat, the first gang consisting of 12 men, the second of 8, the third of 6, and the fourth of 4," [how much does each foreman receive?]

[Adding] 12, 8, 6, [and] 4 makes a total of 30 men. Multiply 30 so as to get 100; the result is 3 1/3; hence the amount for each man is 3 *1/4 1/16 1/64*[100] heqat 1 2/3 ro. Take this amount 12 times for the first gang, 8 times for the second, 6 times for the third, and 4 times for the fourth. [The basic multiplications:][101]

1	3 *1/4 1/16 1/64* heqat 1 2/3 ro
2	6 *1/2 1/8 1/32* heqat 3 1/3 ro
\4	13 *1/4 1/16 1/64* heqat 1 2/3 ro
\8	26 *1/2 1/8 1/32* heqat 3 1/3 ro

[This table is the calculation for the first gang of 12. The same four multiplications are repeated for the next three gangs, but with the respective checking of the fourth multiplication for the 2nd gang, the second and the third[102] multiplications for the 3rd gang, and the third multiplication for the 4th gang.]

[Hence the final] **list of** [the amounts received by] **these** [four] foremen [are tabulated, with] great quadruple-heqat of grain [reduced to regular heqat:]

The first [foreman] with 12 [men receives]
1/4 of 100 heqat + 15, [or] 40 heqat
The second [foreman] with 8 [men receives]
1/4 of 100 heqat + 1 *1/2 1/8 1/32* heqat 3 1/3 ro (or) 26 2/3 heqat
The third [foreman] with 6 [men receives]
20 heqat [*repeated as*] 20 heqat
The fourth [foreman] with 4 [men receives]
13 *1/4 1/16 1/64* heqat 1 2/3 ro [or] 13 1/3 heqat
The total [which the 4 foremen receive is]
100 heqat [or] 100 heqat.

[Problem 69]

3 1/2 heqat of meal is made into 80 loaves of bread. Make known to me the amount of meal in each loaf and their pefsu (*pfšw*)[103] [i.e., cooking potency[184]].

Multiply 3 1/2 so as to get 80.

1	3 1/2
10	35
\20	70
\2	7
\2/3	2 1/3
\1/21	1/6
\1/7	1/2.

The pefsu is 22 2/3 1/7 1/21.

[Proof:]

\1	22 2/3 1/7 1/21
\2	45 1/3 1/4 1/14 1/28 1/42
\1/2	11 1/3 1/14 1/42
[Total :	80].

[3 1/2 heqat makes 1120 ro, for]

\1	320
\2	640
\1/2	160
Total:	1120 ro.

[Hence] multiply 80 so as to get 1120.

The procedure is as follows:

1	80
\10	800
2	160
\4	320
Total: [14]	1120.

So the amount of meal in one loaf [is 14 ro or] *1/32* heqat 4 ro.

[Proof, with the Horus-eye fractions given here in Italics:]

1	*1/32* [heqat] 4 ro
2	*1/16 1/64* [heqat] 3 ro
4	*1/8 1/32 1/64* [heqat] 1 ro
8	*1/4 1/16 1/32* [heqat] 2 ro
\ 16	*1/2 1/8 1/16* [heqat] 4 ro
32	1 *1/4 1/8 1/64* [heqat] 3 ro
\ 64	2 *1/2 1/4 1/32 1/64* [heqat] 1 ro

The result is 3 1/2 heqat of meal [for the 80 loaves, as was specified].

[Problem 70]

7 *1/2 1/4 1/8*[105] **heqat of meal** is made into 100 loaves [of bread]. What is the amount of meal in each loaf and what is their pefsu?

Multiply 7 *1/2 1/4 1/8* so as to get 100.

1	7 1/2 1/4 1/8
2	15 1/2 1/4
\ 4	31 1/2
\ 8	63
\ 2/3	5 1/4
The total is	99 1/2 1/4; the remainder is 1/4
1/63	1/8.

Double the fraction to get 1/4.

1/42 1/126 1/4

The pefsu is 12 2/3 1/42 1/126.

[Proof:]

\ 1	12 2/3 1/42 1/126
\ 2	25 1/3 1/21 1/63
\ 4	50 2/3 1/14 1/21 1/126
\ 1/2	6 1/3 1/84 1/252
\ 1/4	3 1/6 1/168 1/504
\ 1/8	1 1/2 1/12 1/336 1/1008
[Total:	100.]

[Now 7 *1/2 1/4 1/8* heqat make a total of 2520 ro, for

1	320
2	640
4	1280
1/2	160
1/4	80
1/8	40]
Total:	**2520.**

Multiply 100 so as to get 2520:

1	100
10	1000
\20	2000
\5	500
\1/5	20

[Total: 25 1/5, i.e.,] the content of each loaf in ro, which is *1/16 1/64* [heqat] 1/5 ro.

[Proof:]

1	*1/16 1/64* [heqat] 1/5 ro
10	*1/2 1/4 1/32* [heqat] 2 ro
100	7 *1/2 1/4 1/8* [heqat].

[Problem 71]

From 1 **des-jug of beer** 1/4 has been poured off and then the jug has been refilled with water. What is the pefsu of diluted beer?

Calculate the amount of besha (i.e., a kind of grain or fruit) in 1 des of beer; the result is *1/2* [heqat][106] of besha. Take away 1/4 of it, namely, *1/8* [heqat]. The remainder is *1/4 1/8* [heqat]. Multiply *1/4 1/8* [heqat] so as to get 1 [heqat]. The result is 2 2/3, which is the pefsu [of the diluted beer].

[Problem 72]

Example of exchanging loaves for other loaves. You are told that there are 100 loaves of [pefsu] 10 to be exchanged for some number of loaves with [pefsu] 45. [How many of these will there be?]

Calculate the excess of 45 over 10; it is 35. Multiply 10 so as to get 35; it is 3 1/2. Multiply 100 by 3 1/2; it is 350. Add 100 to it;

it is 450. Say then that 100 loaves of [pefsu] 10 are exchanged for 450 loaves of [pefsu] 45, making in wedyet-flour 10 heqat.

[Problem 73]

If it is said to you, "100 loaves of [pefsu] 10 are to be exchanged for loaves of [pefsu] 15. How many of the latter will there be?"

Calculate the amount of wedyet-flour in these 100 loaves; it is [10] heqat. Multiply 10 by 15. This is 150. Reply [then] that this is [the number of loaves for] the exchange.

The procedure is as follows: 100 loaves of [pefsu] 10 would be exchanged with 150 loaves of [pefsu] 15. [It takes] 10 heqat.

[Problem 74]

Another [problem]. 1000 [loaves] of [pefsu] 5 are to be exchanged, [1/2] with [loaves of pefsu] 10 and [1/2] with [loaves of pefsu] 20. What is the exchange of them [i.e., how many loaves of pefsu 10 and how many of pefsu 20 are to be exchanged for each 500 loaves of pefsu 5]?

Evaluate the 1000 loaves with [pefsu] 5. They will take 200 heqat of Upper Egyptian barley. Then say that this is the amount of wedyet-flour in these loaves. Take 1/2 of 200 heqat, namely, 100 heqat. Multiply 100 by 10; it makes 1000, the number [of loaves] of pefsu 10. Multiply 100 by 20; the result is 2000, the number [of loaves] of pefsu 20.

The procedure is as follows: 1000 loaves of [pefsu] 5, made from 200 heqat of wedyet-flour, can be exchanged for 1000 loaves with [pefsu] 10, made from 100 heqat of wedyet-flour, plus 2000 loaves with [pefsu] 20, made from 100 heqat of wedyet-flour.

[Problem 75]

Another [problem]. 155 loaves of [pefsu] 20 are to be exchanged [for a number of loaves] of pefsu 30. [What is that number?]

The amount of wedyet-flour in the 155 loaves of [pefsu] 20 is [to be] 7 *1/2 1/4* [heqat]. Multiply [this] by 30; the result is 232 1/2.

The procedure is as follows. 155 loaves of [pefsu] 20, made from 7 *1/2 1/4* [heqat] of wedyet-flour [can be] exchanged for 232 1/2 [loaves] of [pefsu] 30. [It takes] 7 *1/2 1/4* [heqat].

[Problem 76]

Another [problem]. 1000 loaves of [pefsu] 10 are to be exchanged for a number of loaves of [pefsu] 20 [and the same number] of [pefsu] 30. He (i.e., the learner) shall hear [what that number is].[107]

[One loaf of each kind will take] 1/20 and 1/30 [of a heqat]. [As parts of 30] 1/20 is 1 1/2 and 1/30 is 1. [Added,] the total is 2 1/2.

Multiply 2 1/2 so as to get 30:

1	2 1/2
\ 10	25
\ 2	5

Total: 12.

[Therefore 2 1/2 is 1/12 of 30, so that 1/20 1/30 equals 1/12. Two loaves, one of each kind, will take 1/12 of a heqat and 1 heqat will make 12 loaves of each kind.]

The quantity of wedyet-flour in the 1000 loaves is 100 heqat. Multiply 100 by 12; the result is 1200, which is [the number of loaves of each kind, i.e., for loaves of pefsu] 10 [and those of pefsu] 20. [In summary,]

1000 loaves of [pefsu] 10, making 100 heqat of wedyet-flour can be exchanged for

1200 loaves of [pefsu] 20, making 1/2 of 100 [heqat] and 10 [heqat, totaling 60 heqat of wedyet-flour], and

1200 loaves of [pefsu] 30, making 1/4 of 100 [heqat] and 15 [heqat, totaling 40 heqat of wedyet-flour].

[Problem 77]

Example of exchanging beer for bread. If it is said to you: "10 des of beer [of pefsu 2] are to be exchanged for [loaves of bread of pefsu] 5" [reason as follows to find the number of loaves].

Reckon the amount of wedyet-flour in 10 des of beer; it is 5 [heqat]. Multiply 5 by 5; it makes 25. Say then: "This [i.e., 25] is [the number of loaves for] the exchange."

Proceed as follows:

10 des of beer taking 5 heqat of wedyet-flour can be exchanged for

25 loaves [of bread of pefsu] 5; [for these also take] 5 [heqat of wedyet-flour].

[Problem 78]

Example of exchanging bread for beer. If it is said to you: "100 loaves of bread of [pefsu] 10 are to be exchanged for a quantity of beer of [pefsu]2" [reason as follows to find the quantity of beer].[108]

Reckon the amount of wedyet-flour in 100 loaves of [pefsu] 10; it is 10 [heqat]. Multiply 10 by 2; it makes 20. Say then that this [i.e., 20 des] is [the amount of beer it takes for] the exchange.

[Problem 79]

[Sum a geometrical progression of five terms of which the first term is 7 and the multiplier of each term is 7.]

A house-inventory [shows how to find the multiplication by 7 to find each term as a product in a series].

[Multiply 2801[109] by 7:]

1	2801
2	5602
4	11204
Total:	19607.

[The same procedure is followed to multiply each term in the following series of five numbers by 7, which then may be summed.]

houses	7
cats	49
mice	343
malt	2401 (*corr. ex* 2301)
heqat	16807
Total:	19607.

[Problem 80]

[Horus-eye fractions of a heqat (given on the left in the table below) in terms of the henu, or hinu, as it is often called (given on the right in the table).[110]]

As for vessels (*dbḥ*) used in measuring [grain] by the functionaries of the granary, done into henu (*hnw*):

1 heqat makes 10 henu		
1/2	[ditto]	5 henu
1/4	[ditto]	2 1/2 [henu]
1/8	[ditto]	1 1/4 [henu]
1/16	[ditto]	1/2 1/8 [henu]
1/32	[ditto]	1/4 1/16 [henu]
1/64	[ditto]	1/8 1/32

[Problem 81]

Another reckoning of the henu [from Horus-eye fractions of a heqat].[111]

Now *1/2*	heqat	makes	5	henu
1/4	"	"	2 1/2	"
1/8	"	"	1 1/4	"
1/16	"	"	1/2 1/8	"
1/32	"	"	1/4 1/16	"
1/64	"	"	1/8 1/32	"

[Table a][112]

Now

1/2 1/4 1/8	heqat		makes	8 1/2 1/4	henu		
1/2 1/4	"		"	7 1/2	"		
1/2 1/8 1/32	"	3 1/3 ro	"	6 2/3	"	it is 2/3	of a heqat
1/2 1/8	"		"	6 1/4	"	" " 1/2 1/8	"
1/4 1/8	"			3 1/2 1/4	"	" " 1/4 1/8	"
1/4 1/16 1/64	"	1 2/3 "	"	3 1/3	"	" " 1/3	"
1/4	"		"	2 1/2	"	" " 1/4	"
1/8 1/16	"	4 "	"	2	"	" " 1/5	"

1/8 1/32	"	3 1/3	"	"	1 2/3	"	" " 1/6	"

[Table b][113]

Now

1/8 1/16	"	4	"	"	2	"	" " 1/5	"
1/16 1/32	"	2	"	"	1	"	" " 1/10	"
1/32 1/64	"	1	"	"	1/2	"	" " 1/20	"
1/64	"	3	"	"	1/4	"	" " 1/40	"
1/16	"	1 1/3	"	"	2/3	"	" " 1/15	"

[Table c]

1/32	"	2/3	"	"	1/3	"	" " 1/30	"
1/64	"	1/3	"	"	1/6	"	" " 1/60	"
1/2	"				5	"	" " 1/2	"
1/4	"				2 1/2	"	" " 1/4	"
1/2 1/4	"				7 1/2	"	" " 1/2 1/4	"
1/2 1/4 1/8	"			"	8 1/2 1/4	"	" " 1/2 1/4 1/8	"

[Table d]

1/2 1/8	"			"	6 1/4	"	" " 1/2 1/8	"
1/4 1/8	"			"	3 1/2 1/4	"	" " 1/4 1/8	"
1/2 1/8 1/32	"	3 1/3	"	"	6 2/3	"	" " 2/3	"
1/4 1/16 1/64	"	1 2/3	"	"	3 1/3	"	" " 1/3	"
1/8	"			"	1 1/4	"	" " 1/8	"
1/16	"			"	1/2 1/8	"	" " 1/16	"

[Table e]

1/32	"			"	1/4 1/16	"	" " 1/32	"
1/64	"			"	1/8 1/32	"	" " 1/64	"

[Problem 82][114]

Estimate of the food for a fowl-yard, i.e., the daily portion [of wedyet-flour] made into loaves.

10 birds, i.e., fatted geese, eat [daily] 2 *1/2* [heqat] of wedyet-flour

Taking for 10 days 1/4 of 100 [heqat]
Taking for 40 days 100 [heqat].

The amount of malt that has to be ground to produce (?) it is
1 1/2 times 100 heqat 16 *1/2 1/8 1/32* [heqat] 3 1/3 ro.

The amount of wheat is
1/3 1/4 of 100 heqat 8 *1/4 1/16 1/64* [heqat] 1 2/3 ro.

That which has to be taken away is 1/10 of this, namely,
6 *1/2 1/8 1/32* [heqat] 3 1/3 ro.

The remainder, which is the grain in heqat to be given (i.e., required), is
1/2 1/4 of 100 [heqat] 18 *1/4 1/16 1/64* [heqat] 1 2/3 ro.

Expressed in double heqat this is
1/4 of 100 heqat 21 *1/2 1/8 1/32* [heqat] 3 1/3 ro.

[Problem 82B]

Amount of feed [necessary for other kinds of] fatted birds.

If it takes to fatten 10 birds (i.e., geese) for 1 day 1 *1/4* [heqat],
It will take for 10 days 12 *1/2* [heqat]
And for 40 days 1/2 [of 100 heqat].

The amount of grain to be ground in double heqat is
23 *1/4 1/16 1/64* [heqat] 1 2/3 ro.

[Problem 83]

[Estimate the feed necessary for various kinds of birds.]

As for the feed of 4 geese that are penned up, if it is 1 henu of Lower Egyptian grain [for one day], **the daily portion of feed for one of the fatted geese,** [i.e.,] **the portion** which it eats, is *1/64* [heqat] 3 ro [of Lower Egyptian grain].

As for the feed of a goose that goes into the pond, it is *1/16 1/32* heqat 2 ro; that is, 1 henu for 1 goose.

For 10 geese it takes 1 heqat of Lower Egyptian grain.

For 10 days 10 heqat.

For a month 1/4 of 100 heqat 5 heqat (i.e., 30 heqat).

The daily portion of feed to fatten[115]
A goose is *1/8 1/32* heqat 3 1/3 ro for 1 bird

A tjerp-goose is *1/8 1/32* heqat 3 1/3 ro for 1 bird
A crane (djat-bird) is *1/8 1/32* heqat 3 1/3 ro for 1 bird
A set-duck[116] is *1/32 1/64* heqat 1 ro for 1 bird
A ser-goose is *1/64* heqat 3 ro for 1 bird
A dove is 3 ro for 1 bird
A quail is 3 ro for 1 bird.

[Problem 84]

Estimate the feed of a stall (or stable) of oxen.[117]

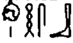
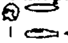
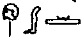
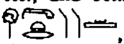
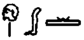
....

4 fine Upper Egyptian bulls eat	24 [heqat]	2 [heqat]
2 fine Upper Egyptian bulls eat	22 [heqat]	6 [heqat]
3 common ...cattle eat	20 [heqat]	2 [heqat]
1 .... ox [eats]	20 [heqat]	
Total of this feed	86 [heqat]	10 [heqat]
It makes in malt	9 [heqat]	7 *1/2* [heqat]
It makes for 10 days	1/2 1/4 of 100 [heqat] 15 [heqat]	1/2 1/4 of 100 [heqat]
It makes for a month	200 [heqat]	1/2 1/4 of 100 [heqat] 15 [heqat]
It makes in double heqat	1/2 of 100 [heqat] 11 *1/2 1/8* [heqat] 3 ro	1/4 of 100 [heqat] 5 [heqat]

[Here ends the mathematical tract in the Rhind Papyrus. The remaining entries, numbered 85-87, are treated and discussed by Chace at the end of his text and translation.][118]

Notes to Document IV.1

[1] Throughout this document the use of bold-faced type indicates rubricated writing in the Rhind Papyrus, as I have noted in some detail in the Introduction to this document. This title in the first line of the treatise (the section given in red in the extreme right-vertical line in Figure IV.2a, Plate 1) has been the object of much attention from the beginning of the modern study of the treatise. The first attempt at translating it was that of S. Birch, "Geometric Papyrus," *ZÄS*, Vol. 6 (1868), p. 109: "Hence the title of the work appears to be the '*principle of arriving at the knowledge of things (or quantities) and of solving*

all secrets which are in the nature of things.' " (The Italics are mine.) Also of great interest is the discussion by F.Ll. Griffith, "The Rhind Mathematical Papyrus," *PSBA*, Vol. 14 (1891), p. 27, where he quotes Vol. 2, Plate 1, lines 1-2 and Vol. 1, p. 27 of Eisenlohr's semiual work (see my Introduction, n. 9): " *Ṭp (? tp) ḥsb n hat (? ḥꜥt) m ḫt: rḫ ntt nbt, snkt (?)* [*nbt, ...nbt*], *štat (?šꜣ?) nbt.*' '*Rule of calculating the results (?) of things*, the knowledge of everything that is, of [all] *obscurities*, [*of all mysteries*] of all difficulties.' The translation of the first phrase and the restoration are both conjectural.  seems to mean, like our 'head' (of cattle), 'an individual,' 'a unit': thence: 'an instance,' 'model quoted as an example'; thence, 'a rule' or 'collection of rules,' as in , , .  in the Bulaq papyrus of accounts is the heading for the balance sheets, and in the *Math. Handb.*, Pl. XX, No. 67, , 'I have calculated' (what was owing).  should therefore mean 'rule of calculation'; , 'rule of the foot,' *i.e.*, of times of entrance, etc., and behaviour; , 'rule of excellence' ? or 'excellent example'; , 'correct example,' or 'rule of correctness.'  and  are often employed in a metaphorical sense, but here the literal meaning of  seems appropriate." Concerning "tep" the cautious Michel Guillemot says ("A propos de la 'géométrie égyptienne des figures'," in *Sciences et techniques en perspective*, Vol. 21 [1992], pp. 126Ter-127): "Pour August Eisenlohr, le premier traducteur du papyrus Rhind, il s'agit d'instructions, de rudiments quand ce ne sont pas simplement des problèmes. Eric Peet distingue seulement 'méthode' et 'exemple'. Plus scrupuleux Arnold Chace traduit toujours par exemple. Enfin Struve oscille entre exemple, forme ou méthode. Per notre part nous sommes tentés d'y voir des 'exemples de'. Mais nous ne devons pas considérer les divers 'problèmes' comme des exemples servant à illustrer une théorie dûment explicitée. Ici, cette dernière est absente, du moins sur le document. Autrement dit 'exemple' doit alors être entendu au sens 'd'action, de manière d'être, considérées comme pouvant être imitées': le mot 'tête' qui sert d'écriture exprime peut être cette possibilité d'imitation. Dès lors les 'exemples' ne sont nullement des règles à suivre impérativement mais des méthodes laissées à un certain libre arbitre."

[2] This is the Hyksos King Apophis, who reigned c. 1585-42 B.C.

[3] This is King Amenemhet III of Dynasty XII, who reigned about c. 1844-1797 B.C. Hence the old copy puts the original text squarely in the very fertile period of the development of mathematical tables and problem collections.
[4] The indications given in brackets that act as titles for the successive divisions of 2 by the odd numbers from 3 to 101 were not in the original text, but the additions make it easier for the reader to find any desired division quickly. However we do find frequently the pertinent denominator appearing by itself at the beginning of the section. For example, note the "5" appearing outside of the bracketed phrase in the first line of the next section covering the division of 2 by 5. Since it is in the text I leave it outside of the square brackets of the subtitle. In his exceedingly nice paper, "The (2:*n*) Table in the Rhind Papyrus," B.L. van der Waerden puts all of this succinctly by saying: "The Rhind Papyrus contains a table, consisting of 50 short sections headed by questioning sentences like 'What part is 2 of 3? What part is 2 of 5? etc. until What part is 2 of 101?' " Actually no such questions specifically appear in the text, but they are implied by the phrase "Call 2 etc." discussed in the next note, and the unit-fraction solutions to the divisions stated prior to the **Procedure** for working them out.
[5] A similar phrase: "Call 2 out of n (where n is one of the successive odd numbers from 3 to 101)" is given at the top of each page of the papyrus with the understanding that it applies to each and every one of the divisions on that page. It is evident that by that phrase the author means to find two to four multipliers of of n that yield products which together sum to 2. The sum of those multipliers in each case constitutes a string of no less than two and no more than four unit fractions solving the desired division under consideration.
[6] This expression means "guidance" to finding the solution already given in the preceding line. As I have explained in the general account of Chapter IV and in the Introduction to this document, the multipliers sought after as unit fractions that will solve the division are singled out by backward slants here in the translation. In the hieratic text, which proceeds from right to left, the marks of selection are forward slants. In fact, the author is by no means consistent in using these slants throughout the papyrus. But it is a simple matter to restore them. Incidentally, another term for "Procedure" (occasionally but not usually rubricated) is used in the calculating tables found in the problems that follow the Table of Two, namely, *irt*, which literally means "the doing" or "the making" (e.g., see Problems 1-6).
[7] Note that the term "**Procedure**" found in the previons division is understood as being pertinent here and in all the succeeding divisions in the Table of Two, whether given or not. Note also that in the fourth line of the working out the author realizes that, for the products of the second column to sum at 2, the next

product must be 1/4. For that to be the case the calculator must determine the multiplier of 1/7 which would produce 1/4. He does this by quadrupling the multiplicand (7) and then taking the reciprocal of that product, namely 1/28. This then will be the multiplier that will be the second unit fraction, which, when added to 1/4, produces the solution of the division sought. This use of reciprocals is repeated again and again in the procedures outlined in the Table of Two.

[8] The Egyptian word which is customarily used in this position is *dmḏ*, which means "total." I have added it in brackets to indicate that, though it is not given in the text here, the following line is the sum of the checked lines.

[9] The Egyptian word is *ḏꜣt*, which means "remainder" or "balance" or "deficiency."

[10] The scribe is careless here. He has left out the "29" and put in its place the "1/24" that should begin the next line. The "1 1/6 1/24" that should follow in the second column has been moved to the third. The scribe placed a check mark before the "1" in the first line of the Procedure, but I have correctly deleted it since the first line is not a part of the solution. Similar carelessness on the part of the scribe occurs often in the text of the Table of Two and has everywhere been noted in the Chace edition; hence I have not given any further notice of it here but have silently accepted the corrections noted by Chace.

[11] These numbers 6, 7, and 5 are placed under numbers 35, 1/30, and 1/42 (see Fig. IV.2e, Plate 10). Note that I have elevated the first number (35) to be a part of the title of the operations, as I have done in a number of cases.

[12] For the necessary insertions of the New York fragments in this and the succeeding divisions of the Table of Two (i.e., 2/89, 2/91, 2/93, 2/95, 2/97, 2/99, and 2/101), see Chace's footnotes to those divisions in Volume 2 of his edition, on the versos of Plates 29-32.

[13] This table concerns the divisions of the numbers 1-9 by the number 10 and Problems 1-6 are specific and practical examples of the divisions of 1, 2, 6, 7, 8, and 9 loaves, successively, among 10 men. In addition to showing how the table can be derived by simple division, R.J. Gillings, *Mathematics in the Time of the Pharaohs*, pp. 121-23, presents an alternative method or rather methods which start with the immediately obvious divisions of 1, 2, and 5, and then proceed to the remaining divisions by the use of addition, doubling, and prior determinations given in the Table of Two.

[14] For the translation of *tp* as "Example" see note 1 to this document.

[15] See the notes relative to Problems 4-9 in Chace's translation, in which he speculates on how the particular doublings given in the procedure or proof tables were obtained.

[16] According to Gillings, *Mathematics in the Time of the Pharaohs*, pp. 109-10: "These 15 problems (including Problem 7B) form in reality a table of 3- and 6-term unit-fraction equalities similar to the Recto Table [of Two in the Rhind Papyrus] and the EMLR [i.e., the Egyptian Mathematical Leather Roll]....." Chace regards them as examples of simple multiplications of fractional expressions, and Neugebauer as completion problems for 2 ÷ 7 and 2 ÷ 9 of the Recto [Table of Two]..... It is my view that this group of problems was included in the RMP to establish a set of 3-term and 6-term equalities for inclusion in Egyptian standard tables, or was taken from a set of such tables." He goes on (pages 110-19) to elucidate this view further. Whatever the ultimate purpose of these problems, I see no reason not to follow Chace's view of them as multiplications of fractional expressions, and I give the expressed multiplications in the lead line of each problem.

[17] Chace in the page facing the text in Vol. 2 has an important note here on what he calls a "curious mistake running through Problems 10, 11, 12, and 14. 1/9 was written as a half of 1/7, and then repeated halving gave 1/18, 1/36, and 1/72. Afterwards someone discovered the mistake and attempted to correct it, but succeeded ouly in part.... In each of the four the total is correct for the numbers obtained by halving from 1/7. It may be that some of these mistakes were made in copying." I have followed Chace's translation in Vol. 1 by giving the correct numbers.

[18] Concerning the mistakes present in the papyrus for this Problem and the preceding one, see the footnotes given by Chace in his text, the versos of Plates 39 and 41.

[19] As Chace points out (on the page facing his plate 44, n. 3), "it is extremely doubtful whether this phrase belongs in this problem."

[20] Chace in his translation of Problem 23 notes that the scribe does not show how to get the fractions 1/9 1/40. Hence, following the procedures of Problems 21 and 22, Chace proffers a possible solution.

[21] These so-called Aha problems are all expressible in modern algebraic form as linear equations (as I have noted below), though the arithmetic techniques of duplication, halving, taking 2/3 and 1/3 of numbers, and taking ten times or 1/10 of numbers, used by the Egyptian and exhibited throughout the Rhind Papyrus are of course employed rather than the conventional modern techniqnes of solving equations. Notice also that a procedure something like that found in later algebras of "false position" is used by the Egyptians, though the proportion that underlies the relations between the false assumption and the true unknown quantity is never specified but rather is ouly implied by the successive multiplications that are carried out by the scribe. The first number assumed to begin the multiplications was in each case obtained by multiplying the denomi-

nators of the fractions of the quantity specified in the enunciations of the problems. I have highlighted this preliminary assumption in the translations of these problems by bracketed phrases which begin thus: "[Assume such-and-such-a-number]." The phrase appears in the Chace translation, but he does not bracket it, which might lead the reader to assume incorrectly that a formal assumption has been specified. The further assumption that is at the heart of the solution of each of the problems is that in the first calculations the value of the unknown quantity, i.e., the Aha, is assumed to be 7 in Problem 24, 2 in Problem 25, 4 in Problem 26, and 5 in Problem 27. The sum of the terms under this assumption must be operated on to produce the actual sum given in the enunciation. The multiples added together to convert the first sum into the enunciated sum yield a number, which, when multiplied by the first assumed number obtained by multiplying the denominators, gives the true value of the unknown quantity. The solution is then checked or proved (so to speak) by adding the quantity songht (and now found) and its fraction or fractions to see if they yield the total specified in the enunciation of the problem. Hence this procedure does resemble the algebraic technique of false position found later in the development of algebra, as I have suggested above.

[22] This is equivalent to the equation $x + \frac{1}{7}x = 19$, without asserting that the Egyptian mathematician solved it with algebraic techniques.

[23] As Chace notes in his translation, the author finds it easier to multiply 2 1/4 1/8 by 7 rather than 7 by 2 1/4 1/8. "A similar change is made in each of the next three problems."

[24] This problem can be expressed algebraically by using the modern equation $x + \frac{1}{2}x = 16$. See the proviso of note 22.

[25] This is equivalent to the equation $x + \frac{1}{4}x = 15$. See the proviso of note 22.

[26] This is equivalent to the equation $x + \frac{1}{5}x = 21$. See the proviso of note 22.

[27] This problem's solution is given rhetorically, without the tabular multiplications found in the preceding problems. Presumably the procedures are the same. Chace adds a possible solution using the Egyptian technique: "It may be supposed that our author first solved the problem as follows:

Assume 9.

\ 1	9
\ 2/3	6

Total	15
1	15
1/3	5
Remainder	10.

As many times as 10 must be multiplied to give 10, that is, once, so many times 9 must be multiplied to give the required number, and therefore the required number [i.e., unknown number] is 9. But now he notices that 9 is obtained by taking away its 1/10 from 10, so he puts in the solution given in the papyrus."

The problem is equivalent to $(x+\frac{2}{3}x)-\frac{1}{3}(x+\frac{2}{3}x)=10.$ See the proviso of note 22. The fact that the phrase "Proceed as follows" is at the end of the problem suggests that the scribe accidentally omitted the calculations involved in this problem when his eye wandered to the calculations used in solving the next problem. This is the suggestion of Peet in his version of the Rhind Papyrus (p. 63). It is given support by the fact that in Problem 29, the enunciation of the problem in the papyrus is missing.

[28] As I have said, the usual calculations are missing.

[29] As I indicated above, the statement of Problem 29 was omitted by the scribe. I have used bold-faced type in the same way that the scribe used rubrication in the preceding problems. This problem is equivalent to this equation:

$$\frac{1}{3}\left[(x+\frac{2}{3}x)+\frac{1}{3}(x+\frac{2}{3}x)\right]=10.$$

Again see the proviso of note 22. It is evident that the calculations given here are only those of the end of the problem where the solution is shown to check with the presumably specified conditions of the problem. Chace in his translation has proposed the preliminary calculations leading to those given in the papyrus: "As in the preceding problem it may be supposed that our author first solved the problem as follows:

Assume 27.

\ 1	27
\ 2/3	18
Total	45
1/3	15
Total	60
2/3	40
1/2	20.

"As many times as 20 must be multiplied to give 10, so many times 27 must be multiplied to give the required number [namely, 13 1/2]. But at this point he seems to have changed the order of these numbers in his mind and to have said, As many times as 20 must be multiplied to give 27 so many times 10 must be multiplied to give the required number.

\ 1	20
1/2	10
\ 1/4	5
\ 1/10	2

Total 1 1/4 1/10

Therefore we must multiply 10 by 1 1/4 1/10 (see Peet, page 64 [of his translation])."

[30] These problems can be expressed by linear equations. The reader will recall that in the preceding group of problems two basic assumptions were made, the first being the assumption of a common denominator for the various fractional coefficients of the unknown quantity so that all the coefficients involving the unknown may be operated on by the usual methods of multiplication and addition that constitute the essential arithmetical procedures of the Egyptians. The second assumption involved an initial "false positing" that the unknown quantity sought is a trial number and then correcting the numerical result of that assumption by finding the multiplier necessary to produce the numerical result given in the initial statement of the problem. While the first assumption is also everywhere present in the solutions of Problems 30-34, the second assumption is replaced by a different form of solution. Since these problems represent divisions, in the Egyptian fashion the author seeks the solution of each of these problems in finding a multiplier that operates on the given multiplicand which consists of the sum of the coefficients specified in the enunciation of the problem in order to produce the whole number specified in the problem. Such a multiplier will then be the desired unknown.

I hasten to note that my account of these methods of solution from an algebraic point of view is somewhat anachronistic, at least so far as the terms I have used (like sum of the "coefficients" of the unknown), terms which owe their origin to algebra. But I stress here that the reader must not deduce from this usage that modern algebraic techniques are necessarily implied. Each problem must be examined on its own to see what techniques may be involved in the solution in each of these implied equations. The distinction between terminology and methods certainly lies behind my discussion of Egyptian Aha problems in Chapter Four. In fact, it appears that, since the problems I now discuss involve collections of a given whole number and fractional parts of an unknown quantity, what is being done in effect by the Egyptians is to factor out, or better add, the collection of coefficients of the unknown, then to consider it as a multiplicand, leaving the value of the unknown as the multiplier. In short, the reader is being asked to find the multiplier (a mixed expression of a whole number and fractions) that will operate on the given multiplicand (which is itself expressed as a mixed number) to produce a given number. Thus all of these problems are in a sense merely arithmetical problems which must be solved by the usual Egyptian procedures of doubling, multiplying by 10, halving, taking 2/3 of, or taking one-tenth of. What is impressive is the manipulative skill of the Egyptians in computing with fractions that is evinced in the solutions of the problems.

[31] The equivalent equation in algebraic form is $\frac{2}{3}x+\frac{1}{10}x=10$. See the proviso of note 22.

[32] In his translation, Chace discusses how the author could have reckoned the remaining 1/30 as 1/23. I need not repeat his statements here, but the reader may find them illuminating.

[33] This is equivalent to the following equation: $x+\frac{2}{3}x+\frac{1}{2}x+\frac{1}{7}x=33$. See the proviso of note 22.

[34] Since this problem, like Problem 30, is a completion problem, it is helpful to add this bracketed phrase to start the solution, a phrase like the one initiating the solution of Problem 30.

[35] In the papyrus this series of numbers appears next to the end of the calculations but in fact belongs here.

[36] This is equivalent to the algebraic equation: $x+\frac{1}{3}x+\frac{1}{4}x=2$. See the proviso of note 22.

[37] See note 34.

[38] In the papyrus this table and the next one are preceded by the bulk of the proof, which was to follow later. The scribe seems to have realized that he wanted these auxiliary calculations before the proof. Hence he added the word "stand" (*ʿḥʿ*), which he then followed with the two tables and the extensive proof I give below.

[39] The equivalent algebraic equation is: $x+\frac{2}{3}x+\frac{1}{2}x+\frac{1}{7}x=37$. See the proviso of note 22.

[40] See note 34.

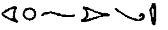
[41] Chace in his translation comments: "The calculation by which the 97 is obtained is given above in the solution of Problem 31."

[42] Concerning this and the preceding fraction Chace in his translation says: "We may notice that in the course of his proof our author has 2/3 of 1/679 equal to 1/1358 1/4074, a remarkable application of the rule given in Problem 61[B]."

[43] This is equivalent to the equation: $x + \frac{1}{2}x + \frac{1}{4}x = 10$. See the proviso of note 22.

[44] Problems 35-38 were solved essentially by the method that resembles algebraic false position, which we have noted appears to have been used in Problems 24-29. It was also used in Problems 40 and 70. I alert the reader to the statement made by Chace in the first volume of his edition of the Rhind Papyrus, p. 11, where he further notes that Problems 35, 37, and 38 "show more clearly [than either of the preceding groups of problems couched in terms of 'quantity' only] that this process enables us to keep in mind the nature of the quantities involved." He then goes to explicate in some detail this statement by analyzing what he believed to be the reasoning of Problem 35.

[45] Chace in his free translation says about the enunciations of problems 35-38: "In these problems in the papyrus the questions are put in a curious way: 'I have gone a certain number of times into the *hekat*-measure, certain parts have been added to me, and I return filled. What is it that says this?' It is stated as if the vessel represented as speaking had gone into the *hekat*-measure and returned filled, but clearly it is the *hekat*-measure that is filled."

[46] See Griffith's "Notes on Egyptian Weights and Measures," *PSBA*, Vol. 14 (1892), p. 426, and also A. Gardiner, *Egyptian Grammar*, 3rd ed. (revised, London, 1973), pp. 197-99, and his marginal citations. See also Fig. IV.3 for the diagram of the Horus eye with the indication of the parts of the Horus-eye and their fractional values. As that diagram indicates, the whole eye is represented by the following glyph: . The parts of the heqat-measure are represented by parts of the glyph, i.e., 1/2, 1/4, 1/8, 1/16, 1/32, and 1/64 successively by these parts of the glyph: .

[47] The heqat-measure is not mentioned in this problem, but it is clear that this addition is understood since the problem obviously belongs to this group of problems.

[48] See the enunciation of Problem 35. Here I have silently made the same change since this change fits the meaning and the working out of the solution of the problem.

[49] This is mistakenly written as 80 in the papyrus.

[50] This is mistakenly given as 1/265, i.e., 265 with the fractional dot over it. Furthermore, it is written in black rather than red, but I have rendered it in boldface to help the reader.
[51] Chace in his translation (Vol. 1, p. 81) discusses the procedure and its peculiarities.
[52] See Chace, *ibid.*, p. 83, for a discussion of the numerical peculiarities of this problem.
[53] This is a simple division problem, namely, (50÷4)–(50÷6)= x. As the author shows, the excess or difference of share for the group of 4 over the group of 6 is 4 1/6 loaves.
[54] As Chace notes *(Ibid.*, p. 84*)* this problem is so simple as to be hardly worthwhile. He goes on to say: "It is possible that the author intended to state a problem in arithmetical progression like the next one." He also remarks that "he (the author) considers the [group of] 4 first in the solution, although he puts the [group of] 6 first in the statement."
[55] As I have already remarked in note 44, this problem uses the technique of false position to initiate the solution.
[56] See the discussion of the volume of a cylinder in the section "Volumes" in Chapter Four. It is evident that the volume of the cylinder is determined in this problem by multiplying the area of the circular base by the altitude of the cylinder. It is also obvious that the area of the circle of diameter 9 is assumed to be the square of side 8, i.e., 1/9 of the diameter is subtracted from it, leaving 8, which is then squared. Looking at the other problems where the area of circles is determined (for example see the next problem), the area of the circle is always calculated by subtracting 1/9 of the diameter (whatever it is, though it is usually posed to be 9) and squaring the result.
[57] This is exactly like the preceding problem except that the diameter is 10 instead of 9, and thus involves the solution in many fractions. Notice that this figure is written in the papyrus as 5900 heqat 1/4, the latter equates to 5900 heqat 25. Indeed we note that in the text there is no computation of the multiplication of 1/108 times 100. See Chace's translation (*ibid.*, p. 87) and his note 2 to text of Problem 42 (Vol. 2 , on the page facing Plate 64), for remarks on the incompleteness and irregularities of calculation and notation here.
[58] See the discussion of the volume of a cylinder in khar in the geometric section on Volumes in Chapter Four. This problem, which puzzled Egyptologists in the form that exists in the papyrus, will first be given as in the papyrus, and then followed by two reconstructions. While both are plausible, the second one, constructed by Gillings, preserves much more of the text as found in the papyrus. It is obvious that the purpose of the problem is to present an example of a method for directly finding the content or volume of a cylindrical granary in

khar without first finding it in cubic cubits. Both reconstructions achieve that end. Chace (Vol. 1, pp. 88-89) discusses Eisenlohr's futile efforts to understand this problem in its erroneous form.

[59] These are given as 1/2 and 1/4. But since they are fractions of 100, they in fact equal the amounts I have given here in the translation.

[60] Cf. T.E. Peet, *The Rhind Mathematical Papyrus* (London, 1923), pp. 82-83 and Chace, *op. cit.*, Vol. 1, p. 88. The method of finding the volume of a cylindrical granary in khar directly without finding it first in cubic cubits, which is the object of Problem 43 despite its textual confusions, is the same as that of a problem in a fragmentary papyrus (Kahun, IV. 3) rendered in Document IV.3 below and discussed in the section on the volume of a cylinder in Chapter Four. This reconstruction here in the second version of Problem 43 has the great difficulty of discarding the whole column of reckoning found in the papyrus, a fault so grave as to cause us to reject this reconstruction. The pertinent brief note of H. Schack-Schackenburg is "Die angebliche Berechnung der Halbskugel," *ZÄS*, Vol. 37 (1899), pp. 78-79, where he notes that these data in the Kahun fragment interpreted by Borchardt as concerning a hemisphere were in fact concerned with the volume of a cylinder with a "simplified" *(vereinfachte)* calculation like that of Problem 43 in the Rhind Papyrus, "dabei aber irrthümlich die bei früheren Rechnungsweise erforderliche Subtraction von 1/9 des zu quadrirenden Durchmessers beibehalten, weshalb er nur 64/81 des richtigen Resultats erhält."

[61] R.J. Gillings, *Mathematics in the Time of the Pharaohs* (Dover ed., New York, 1982), pp. 148-51. Gillings also sees the purpose of the problem to find the volume of the cylinder in khar directly, he would make two essential changes in the initial statement, first to change the diameter from 9 to 8, and the second to remove the sentence and a half in the Rhind text that reads: "To take away 1/9 from 9; the remainder is 8. Add to 8 its 1/3"; and insert instead: "Add to the diameter its 1/3." This allows him to preserve all of the calculations added to the procedural paragraph in the Rhind Papyrus. Hence his presentation is the more economical one and is to be preferred to the two preceding versions.

[62] This problem is the reverse of Problem 44.

[63] Chace, Vol. 1, p. 90, adds: "Instead of taking 2/3 at the beginning to reduce the contents to cubed cubits, as he would have done if he had exactly reversed the process of the preceding solution, the author takes 2/3 of the last quotient to find the last dimension."

[64] There is here in the Rhind Papyrus a crude drawing of an octagon circumscribed by a square, as the reader can see in Fig. 2ii, Plate 70. In Chace's translation he interpreted the problem as the comparison of a circle of diameter

9, whose area was considered to be equal to $(9-1)^2$, i.e., 64, which reflects a formula found in the granary problems 41 and 42, and also in area Problem 50. K. Vogel, *Vorgriechische Mathematik,* Teil 1 (Hannover, 1958) and, after him, Gillings *(op. cit.,* p. 142) interpret the problem as a comparison of an a symmetrical octagon composed of five smaller squares of side 3 plus 4 half such squares and the octagon's circumscribed square of side 9. Vogel notes, however, that the area of such a square would be 63 not 64, but he considered that a good enough approximation. I am inclined to accept his view that we are concerned with a square equal to the octagon which itself is approximately eqnal to the inscribed circle, but I am doubtful about this interpretation of the numerical approximation of octagon and square, because the numbers actually in the papyrus do not bear him out. The alternative method of showing graphically the rough equality of octagon and circle and then computing the approximate equality of the octagon to a square of side 8 proposed by Gillings (pp. 143-46) is somewhat more interesting. See my discussion of this solution and the perhaps better one of Guillemot in the section on areas in Chapter Four.

[65] A setjat (*sṯ3t*) or setat (as Chace transliterates it) was a square khet, while a khet was a unit of length of 100 cubits. See note 76 below.

[66] Chace (Vol. 1, p. 92) explains the calculation as follows: "The papyrus states the problem for a field of 10 khet by 2 khet, and these numbers are in the figure, but the solution is for 10 khet by 1 khet, or 1,000 cubits by 100 cubits. Multiplying these numbers together gives 100,000 square cubits [as in the third line of the calculations]. Dividing this by 100 gives 1,000 cubit-strips [as in the last line], strips 1 cubit wide and 1 khet long."

[67] This is the conventional method of squaring a circle among the Egyptians, namely, subtracting 1/9 of its diameter, and then squaring that remainder, as we have explained in Chapter Four in the section on geometry.

[68] There are contrary views of whether "side" or "height" is meant here. See my discussion of the area of a triangle in Chapter Four under the rubric "Ancient Egyptian Geometry: Areas." But let me add a few remarks here. If "height" is meant, as I feel certain, then this is the conventional formula of A = 1/2 base times altitude (with simply the base and altitude interchanged). This is my view and that of Gunn and Peet, Struve, Gillings, Couchoud, and others (see Gillings, *op. cit.,* pp. 138-39). On the other hand, if it means "side," this makes the given calculation inaccurate. Chace (Vol. 1, p. 36) expresses this view as follows: "In Problems 51-53 the Egyptian determines the area of a triangle by multiplying 1/2 of its base, and the area of a trapezoid by multiplying 1/2 the sum of its bases, by the length of a line *(meret)* which, so far as our present knowledge goes, might be either the side or a line representing the altitude. In the latter case he would be correct. In case the triangle is isosceles with a nar-

row base as compared with its height, he would be nearly correct, even if *meret* means side. Personally I am inclined to think that this word does mean side in geometry, and that the author intended to consider only isosceles triangles with narrow bases. In Problem 51 the base is comparatively narrow, 4, with *meret* equal to 10." (See also *ibid.*, pp. 132-34.) M. Guillemot, "A propos de la géométrie égyptienne," pp. 129-35, after careful treatment of the various terms used in the problems devoted to triangles in both the Rhind Mathematical Papyrus and the Moscow Mathematical Papyrus, seems to conclude that Problem 51 of the former and Problem 4 of the latter do concern an isosceles triangle and that the area of each as determined by the multiplication of one of the equal sides by half the base is an acceptable approximation, but that the triangles given in Problems 7 and 17 in the Moscow Papyrus are right triangles, the area of each of which is determined by 1/2 the product of the two sides containing the right angle. But he is by no means dogmatic in his conclusions.

Off hand, it would surprise me that the author of Proposition 51 would use the "side" as one of the multipliers in this fashion, since he was sophisticated enough in the next problem to take 1/2 of the sum of the bases of a trapezoid to find one side of the equivalent rectangle, but there are better arguments than this. We have to consider that the author is thinking of converting the triangle to a rectangle, where the computation is in terms of the sides of a rectangle, a rectangle that must be half the size of a rectangle with the same height and base as the triangle (as is evident from physical inspection). All of this is discussed in Chapter Four above, as I have already indicated.

Of those views that support the conclusion that the meaning of *meryt* in the determinatious of triangular areas is "height," the most cogent to me is the one based on good philological reasoning, first proposed by B. Gunn in his review of Peet's edition and translation of the Rhind Papyrus in *JEA*, Vol. 12 (1926), p. 133, and then more briefly but more pointedly by Gunn and T.E. Peet in their "Four Geometrical Problems from the Moscow Papyrus," *JEA*, Vol. 15 (1929), p. 173 (full article, pp. 167-85): "In the *Rhind Papyrus*, and in *Moscow Pap.*, Nr. 4, what we call the 'height' of a triangle is called the *emröyet (mrjt)*, a word meaning, among other things, a 'quay' erected on a river bank. A glance at Fig. 1 (=my Fig. IV.4a) will show how appropriate the term is; the 'upper' side AB of the triangle *ABC* appears as the sloping river-bank, and the *emröyet* is the horizontal quay above it." For *mryt* as a quay, see *Wb*, Vol. 2, p. 110 (top). Also see E.A.W. Budge, *Egyptian Hieroglyphic Dictionary*, p. 308a, and Peet, *The Rhind Mathematical Papyrus* (London, 1923), p. 91. See also S. Couchoud, *Mathématiques égyptiennes*, pp. 46-48.

W.W. Struve, *Mathematischer Papyrus des Staatlichen Museums der Schönen Künste in Moskau* (Berlin, 1930), pp. 123-34, discusses the similar problems on the area of a triangle in the Moscow Papyrus, with this conclusion that the Egyptians did indeed compute the area of a triangle as 1/2 the base times the altitude, holding this view against many of the early Egyptologists (see below Document IV.2, Problems 7 and 17) and my discussions of triangles in the section on geometry in Document IV.2.

For the three possible triangular figures being computed in Problem 51, and thus being discussed in this note, see Fig. IV.4b. For a possible graphic discovery of the area of a scalene triangle, see Fig. IV.4c based in part on Problems 7 and 17 in Document IV.2.

Notice further that in this problem the calculation precedes the general statement in abnormal fashion.

[69] Chace (Vol. 1, p. 92) explains the calculation as follows: "In the reckoning he (the author) puts down the base as 400 and the side [height?] as 1000; that is, he expresses these lengths in cubits. Dividing 400 by 2 he gets 200 and 1000 as the dimensions of the equivalent rectangle. Then to obtain the area expressed as so many cubit-strips he multiplies 1000, not by 200, but by 2, as if he thought of the rectangle as made up of 1000 pairs of cubit strips. Finally, he writes down 2, that is, 20 *setat* (2 ten-*setat*), as the standard form of expressing the result."

[70] The papyrus incorrectly has "20." The phrase "100 setjat" appearing in the translation is an interpretation of the "10 ten-setjat" in parentheses. Notice that again I prefer to render *mryt* as "height" by the same argument that I have given in note 68. See also the discussion of Problem 52 in the geometry section of Chapter Four.

[71] The enunciation of the problem is missing. Hence we must deduce the objectives of the problem from the two sets of calculations and the drawing that appear in the papyrus. The drawing given in Fig. IV.2LL is reproduced and enlarged in Fig. IV.5a, along with a copy of it having the hieratic numbers translated into modern numerals. Clearly the figure as drawn is an isosceles triangle. But the numbers given on the figure, if correct, make it impossible for the figure with its three sectious to be an isosceles triangle. For example, the two bases of the trapezoidal section to the far right cannot both be "6" as they are so marked, for then the trapezoid would be a rectangle instead of a trapezoidal section of the isosceles triangle. Similarly, the next trapezoid (in the middle of the figure) with its two bases marked as "6" and "2 1/4" cannot be a part of the whole triangular figure drawn as isosceles, if the isosceles triangular section on the left has a base of "2 1/4" and an altitude of "7" as so marked on the drawing.... Hence if all the numbers are correct then the whole figure must not be an isosceles triangle but rather a three-tiered figure compounded of a rectangle, a trapezoid, and a triangle (see Fig. IV.5b). But even this unlikely

conclusion stumbles over the errors in the first set of calculations, which we assume to be the middle trapezoidal section, and see the next note. Incidentally, Chace in his translation (Vol. 1, p. 94) interprets the trapezoidal-triangular figure that is the object of this problem as an isosceles triangle and the "sides" of the trapezoidal sections and of the triangular section as actually being sides rather than altitudes. See my addition of Chace's version of the first set of calculations following Problem 53 in the text and Fig. IV.5c. This allows him to reinterpret and alter the first set of calculations in the manner that he interpreted the calculations of the triangle and trapezoid in the two preceding problems. Thus the areas determined in his reconstruction are in error in the same way as the triangle and trapezoid of the previons two problems. See notes 74 and 77 below for further comments on Chace's reinterpretation.

72 Though we are not completely sure whence the number 4 1/2 came. It seems likely that it was a mistake for 1/2 of the sum of the bases of the trapezoid, which in fact equals 4 1/8 rather than 4 1/2. But once having used 4 1/2 in his multiplications, he multiplied by the altitude (3 1/4) to get the first approximation to the trapezoidal area, which, however, he incorrectly computed or by a slip of concentration gave 5 1/4 1/8, adding lines 1 and 4 of his calculating table rather than lines 1, 2, and 4, which would have given the correct figure of 14 1/4 1/8. But remembering that he (or more likely the original author) had taken a larger top base line than he should have, he corrected it by first taking 1/10 of the correct total (which was perhaps in his original copy) and then subtracting that 1/10 from the correct total. This took him fairly close to the correct area of the middle trapezoidal section. See the next note for details.

73 If we gave this figure in setjat and its fractions, we would see that 1/10 of the total (14 1/4 1/8) would be 1 1/4 1/8 1/16 1/40. But notice that the last two fractions are approximated as 10 cubit strips (instead of 10 3/4 cubit-strips). I remind the reader that the "cubit-strip" is 1 khet or 100 cubits long and 1 cubit wide. The term "cubit-strip" is Chace's (Vol. 1, p. 33). He adds to the definition: "100 cubit strips make a *setat*." Hence the area of the trapezoid determined in this fashion would be about 10 cubit-strips less than 13 setjat. But if we computed the area using the correct half of the sum of the two bases, i.e., 4 1/8, without needing to subtract anything, and multiplying that by the altitude 3 1/4, the area would be 13 *1/4 1/8 1/32* setjat.

74 The reader should remember that Chace believed that the "7" was the side of the triangular element and not its altitude. However, in giving his description of the lower trapezoid he speaks of 3 1/2 as "the height or side" and presumably he would have made the same concession for the small triangle. If "7" is the side and not the height, then the area computed is, of course, not the true area and in fact is not a very good approximation.

[75] See notes 71, 74, and 77. See also the comments I have added in brackets below this text of Chace's proposed correction of the first set of calculations.
[76] The numbers I have given in Italics are given in boldface type by Chace in his translation. Needless to repeat, I have usurped boldface type throughout my version of the translation in order to represent rubrication used by the scribe of the papyrus. Hence, my use of Italics for the signs for Horus-fractions and other numbers which are given by Chace in boldface type.
[77] See Vol. 1, pp. 93-94.
[78] The bracketed alternative is given by Chace in his translation (Vol. 1, p. 95). Commenting on this problem and its successor Chace says (*ibid.*, p. 96) the following: "These two problems are simple division problems–10 is multiplied so as to get 7 and 5 so as to get 3–and they have been translated both by Eisenlohr and by Peet, 'Divide...into...fields.' But the preposition [*ḫnt*] sometimes means *from* and does not mean *into*, and the verb at the beginning [*ḫbt*], which is used several times in the papyrus, elsewhere always means *take away* or *subtract*. Gunn ['Notices of recent publications,'] (page 133) has suggested that these words can be used here with their ordinary meanings in the sense of taking away an equal part from each field.

"In each of these problems a product and multiplier are given to find the multiplicand. Problem 54 is, How large a field taken 10 times (once from each of the given 10 fields) will make 7 *setat*, and Problem 55, How large a field taken 5 times will make 3 *setat*? As the Egyptian cannot solve these problems directly, he forms new ones in which the multipliers in these become multiplicands and the answers are obtained first as multipliers....In writing down the multiplications of these new problems he writes all of his numbers as mere numbers, but in Problem 55 he writes first the statement of his new problem as a problem in *setat*. The answer to this new problem is 1/2 1/10, and if it is taken as a problem in *setat*, the argument for the answer to the given problem will be, 1/2 1/10 times 5 *setat* makes 3 *setat*, therefore 5 times 1/2 1/10 of a *setat* (or as he has to write it, *1/2 setat* 10 cubit-strips) will be 3 *setat*, and so the answer to the given problem is *1/2 setat* 10 cubit-strips." [The fractions of setjat in this answer is written by Chace in boldface, and I have written them in Italics, following my convention.]
[79] Instead of the number "5" the papyrus has "1 setjat."
[80] Again, I take the bracketed phrase from Chace. For an explanation of this problem, see note 78. At this point the text literally says: "Perform the operation on 5 setjat for finding the area of 3 setjat."
[81] This phrase occurs following the literal sentence quoted in note 80.

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[82] As I noticed in Volume Two of my work (p. 76), the Egyptians not only used the concept of slope in the determination of pyramids but also in their conical water clocks.

[83] The slope is here the horizontal length for every cubit rise in height. It is thus equivalent to the cotangent of the base angle of the faces of the pyramid whose altitude is 250 cubits and whose base side is 360 cubits, i.e., the cotangent of the base angle of the right triangle whose altitude is 250 cubits and whose base side is 180 cubits. The solution is given in palms, where 1 cubit = 7 palms. See my brief discussion of this problem in Chapter Four under the rubric "Volumes." The pyramids found in these problems are all right pyramids. Notice that the  in seqed is rubricated. Note further that among the early historians the transliteration of the term was usually "seked."

[84] The pyramid in this problem (when its measurements are converted to feet) has a base-side of 618 feet and a height of 429 feet. The original height of the Great Pyramid of Cheops at Giza was 481.4' and its base-side averaged 755.8'. See Gillings, *Mathematics in the Time of the Pharaohs*, p. 185 and I.E.S. Edwards, *The Pyramids of Egypt* (Harmondsworth, England, repr. 1975), p. 118.

[85] This problem is the inverse of the preceding one. Chace in his translation (p. 97) says that here and in Problem "59B the author doubles the *seked* instead of taking 1/2 of the side of the base, and instead of dividing the *seked* doubled by 7 and dividing the side of the base by the result, he divides 7 by the *seked* doubled and multiplies the side of the base by the result, which amounts to the same thing."

[86] For the interchanging of numbers in the text, see Chace's translation (p. 98).

[87] This problem is the inverse of the preceding. See *ibid.*

[88] The word appears to be that of a pillar, but Peet, *op. cit.*, pp. 100-02 suggests a " cone" as a possibility and has a long discussion of the figure involved. As I have already noted the ancient Egyptians were surely interested in inverted conical water clocks and their slopes to find an even flow of water therefrom (see note 82 above). If a cone is intended, then the expression "sentjet" for base would have to be the "diameter" and thus its 1/2 length the "radius."

[89] Note that in the triangular figures of Plate 82, the base in each case is a single line rather than the double base-lines of the pyramids in the illustrations of the preceding problems. This might be regarded as support for considering Problem 60 as being concerned with a cone rather than a pyramidal column or pillar.

[90] Chace in his translation (p. 99) says: "In the papyrus, instead of dividing the base of the right triangle by the other given line, the author divides the other given line by the base. I follow Borchardt (1893) in treating this as a mistake. At the end he does not multiply by 7 so as to express the *seqed* in palms, as he

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[86] For the interchanging of numbers in the text, see Chace's translation (p. 98).
[87] This problem is the inverse of the preceding. See *ibid.*
[88] The word appears to be that of a pillar, but Peet, *op. cit.*, pp. 100-02 suggests a " cone" as a possibility and has a long discussion of the figure involved. As I have already noted the ancient Egyptians were surely interested in inverted conical water clocks and their slopes to find an even flow of water therefrom (see note 82 above). If a cone is intended, then the expression "sentjet" for base would have to be the "diameter" and thus its 1/2 length the "radius."
[89] Note that in the triangular figures of Plate 82, the base in each case is a single line rather than the double base-lines of the pyramids in the illustrations of the preceding problems. This might be regarded as support for considering Problem 60 as being concerned with a cone rather than a pyramidal column or pillar.
[90] Chace in his translation (p. 99) says: "In the papyrus, instead of dividing the base of the right triangle by the other given line, the author divides the other given line by the base. I follow Borchardt (1893) in treating this as a mistake. At the end he does not multiply by 7 so as to express the *seqed* in palms, as he

does in the other problems." I point out here that what the author has done by his so-called mistake is to express the seqed as the tangent of the base angle rather than as the cotangent.

[91] The table is not a part of the text. For the comments of Peet and his own remarks concerning this table, see Chace's translation (pp. 24-25, 100) and his text (Vol. 2, page opposite Plate 83).

[92] Chace (Vol. 2, page opposite Plate 84, n. 10) notes that the text has "shaty," but that it ought to have "deben." Incidentally, shaty (meaning "seal") is a unit of value and deben a unit of weight. 1 deben is about 91 grams.

[93] It is clear the procedure for finding the answer here is like that of algebraic "false position" used in "aha" problems (see note 21 above).

[94] The procedure given here for finding the terms of an arithmetic progression is essentially the same as that of the modern formulation: $h = (S/n) + (n-1)(d/2)$, with h the highest term, S the sum of the terms, n the number of terms, and d the common excess. See my discussion of this problem in Chapter IV.1 under the rubric "Arithmetic and Geometric Progressions." A similar problem is found in the Kahun fragments, Cols. 11-12, which see in Document IV.3, and consult especially note 1 of that document. Notice that the fractions of the heqat of barley given here are the so-called Horus-eye fractions and hence, as everywhere, I render them here by Italics.

[95] The solution is again equivalent to the procedure of false position.

[96] Here is a convincing statement that *rnpt* was often (and no doubt, usually) used for the whole civil year, i.e., for the 360 days of the months plus the 5 epagomenal days. See the discussion in my *Ancient Egyptian Science*, Vol. 2, pp. 177-78, n. 2.

[97] I have added the checks to indicate the pertinent entries.

[98] As in Problem 61B, this is a specific indication that the problem and its solution show us how to do any similar problem. This indeed must be the purpose of most of the problems, namely, to give a model solution of a particular kind of problem.

[99] Note I have added "[with]" because in fact the problem does not have the objective of determining the value of the tribute-cattle, but rather with determining the size of the herd that produces the specified number of tribute-cattle. Counting of the cattle of the land, presumably for tribute or tax purposes, goes back to the early dynasties and is mentioned in the Early Egyptian Annals (the Palermo Stone), as I have noted in my *Ancient Egyptian Science*, Vol. 1, pp. *51-52.*

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[101] As Chace says in his translation (p. 105): "This problem is merely to divide 100 into four parts proportional to the numbers 12, 8, 6, and 4. The laborious numerical calculations are due to the use of 'Horus-eye' fractions."
[102] Actually there is no check before the third multiplication, but obviously one was intended since the addition of the products produces the correct answer.
[103] As Chace reports in a note to his text (n. 2, page opposite Plate 91): "The stem was originally *fśw*. Later the pronunciation of the first consonant was probably changed to *p* and a *p* was added, although the *f* was retained in writing. Late writings show *pśw*, with the unpronounced consonant abandoned." (I have used the transliterated letters *p* and *f* for Chaces "p" and "f".) See the use of pesu in the Moscow Papyrus (Doc. IV.2, Problems 15 and 24) and the remarks of its editor W. W. Struve, *Mathematischer Papyrus des Staatlichen Museums der Schönen Künste in Moskau* (Berlin, 1930), pp. 45-49.
[104] This term indicates inversely the strength of bread or beer after it has been cooked or brewed. Pefsu is the ratio of the number of loaves of bread or jugs of beer produced to the number of heqat of grain used to produce them. Hence the higher the pefsu number the weaker the product. Chace in his translation (p. 105) says that "It meant something like 'cooking ratio,' that is, the number of units of food or drink that conld be made from a unit of material in the process of cooking, and it determined the relative value of any food or drink.... We may note that the lower the *pefsu* the more valuable the unit of food." Notice that the author solves first for the pefsu and then for the quantity of meal in each loaf; hence he answers the second question first.
Problems 69-71 concern the finding of pefsu, while Problems 72-78 are devoted to reckoning the exchange of loaves of bread of differing pefsu or of beer and bread of differing pefsu. See my discussion of these various problems in Chapter Four under the rubric "Pefsu Problems."
[105] Again notice the use of the Horus-eye fractions, represented here by Italics.
[106] This first writing of 1/2 is mistakenly given in ordinary fractional signs; but, as Chace points out, it should have been written with a Horus-eye sign; and so I have changed it to Italics and added "[heqat]," as I have done elsewhere.
[107] Gillings shows how this problem could have been solved by the finding of the harmonic mean or average of the pefsus of the equal number of loaves for which the original 1000 loaves were to be exchanged (*op. cit.*, p. 131). But it is clear the arithmetic solution involving the determining of the harmonic mean and its use is not quite like that by which the scribe did solve it.
See my analysis of the problem in Chapter IV under the rubric "Pefsu Problems.(pp. 65-66)."
[108] This problem reverses the direction of the exchange, namely, to find the quantity of beer to exchange for the 100 loaves of bread.

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[110] While the Horus-eye fractions are noted in the parts of the eye as always, the henu are given by the regular fractional signs.

[111] The first part is the same as the table in Problem 80, namely the figures for the henu-equivalents of the Horus-eye fractions of a heqat. It should be noted that the heqat measures of the henu never use the Horus-eye glyphs. Nor, in fact, in the third columns of tables a–e, are fractions of a heqat ever written with Horus-eye signs, but they are always given there in regular fractional signs–even when fractions are 1/2, 1/4, 1/8, 1/16, 1/32, and 1/64 in the first column of each table.

[112] Chace discusses the various tables and notes that there are a few errors and difficulties with the scribe's writing; he has made the corrections that seemed most probable.

[113] Notice that the fractions of a heqat given in the third column of each of the tables b–e are rubricated, and hence I have as usual given them in boldface.

[114] Problems 82-84 concern the calculation of the amount of food necessary for birds and oxen. Chace gives an interesting evaluation of the problems on the feeding of the birds on page 116 of his translation.

[115] The following birds are discussed by Peet, *op. cit.*, p. 126.

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[117] The rest of the first line in the papyrus is difficult and unintelligible. In fact the whole problem is full of difficulties. See Chace's text and translation and Peet's remarks on the same problem.

[118] While I shall not treat of these additions, I note that the fragment in Number 87 includes the epagomenal days (i.e., at least the third and fourth of the epagomenal days celebrating the Births of the Gods Seth and Isis) at the head of the year, i.e., as the first part of the first month of the season of Akhet: (Chace's

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translation, p. 119) : "Year 11, first month of the inundation season, third day, Birth of Set; the majesty of this god caused his voice to be heard. Birth of Isis, the heavens rained." This bears on the discussion in Volume Two of my work (pp. 178-79) concerning the positioning of these days at the head or at the tail of the year.

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DOCUMENT IV.2

The Moscow Mathematical Papyrus: Introduction

The second most important mathematical document that will help the reader understand the character and achievements of ancient Egyptian mathematics is the mathematical papyrus of the Museum of Fine Arts in Moscow. It is numbered 4676 in the inventory and is part of the collection of W.S. Golenischeff (also transcribed as V.S. Golenishchev), and the first complete edition of its 25 problems was that prepared by W.W. Struve in 1930.[1] Golenischeff bought the papyrus in 1892/3 or 1893/4, telling us the circumstances in a reply written in 1929 to a request for information (concerning its discovery and purchase) from L.S. Bull:[2]

> En réponse à la demande, que vous m'adresser, je ne puis malheureusement vous donner qu'un bien maigre renseignement. À un voyage, que je fis en Égypte, si je ne me trompe, en 1892/3 (ou bien en 1893/4), j'ai eu l'occasion d'acheter le petit papyrus mathématique chez Abd el-Rasoul, un des frères, qui autrefois avaient détenu le secret de la cachette royale de Deir el-Bahari. C'était, si je m'en souviens bien, l'ainé des frères, notamment celui qui après une bonne bastonnade, avait dévoilé le secret, du temps de Mr. Maspero, et qui, ayant plus tard reçu, pour le petit dérangement subi, une somme d'argent de la part du Gouvernement Égyptien, s'était bâti une maisonnette au pied de la colline de Sheikh Abd el-Qourna. Un jour, en revenant d'une visite aux tombeaux de Qourna, je m'arrêtai chez Abd el-Rasoul, que je connaissais de longue date, et c'est lui qui, au moment des adieux, m'offrit pour une somme assez modique ce petit manuscrit. Lorsque je m'en rendis acquereur, le papyrus n'était pas encore déroulé et c'est en

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From this account it is evident that the papyrus came from a tomb not too far from the place where the Rhind Papyrus was discovered.[3] As the editor in the introduction to his translation and commentary has shown by analyzing the paleography and orthography, it is quite likely that the small Moscow Papyrus was written down in Dynasty 13 and was dependent on some work (perhaps of a different nature) written in Dynasty 12.[4] Hence the version presented here is perhaps not too far removed in time from the "ancient copy" which was the source of the Rhind Mathematical text.

The first mention of the Moscow Papyrus was that by M. Cantor in the second edition of his *Vorlesungen über Geschichte der Mathematik*, Vol. 1 (Leipzig, 1894), p. 23, who merely notes that, in addition to the Rhind Papyrus, there was a mathematical papyrus, which belonged to Wladimir Golenischeff. The first attention to the contents of the papyrus came in 1917 from B.A. Turaeff (Turaev), the conservator of the Egyptian section of the Moscow Museum.[5] It dealt principally with Problem 14 of the papyrus (then numbered 9 since the traces of problems found in accompanying fragments were not counted) and concluded that the Egyptians used a formula for the volume of a truncated square pyramid equivalent to the multiplication of 1/3 of the altitude by the combined sum of the areas of the two bases and the square root of their product, i.e.,

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Before discussing the nature of the work given in the Moscow Papyrus and comparing it with the mathematical work in the Rhind Papyrus, let me list the problems in the order given in the papyrus. In doing so, I follow the text of Struve,[7] which is based on his interpretation of the full problems and fragments appearing in the papyrus (so that often, as in the first two problems, the fragments are so slight that their reconstructions by the editor are entirely speculative, as will be clear to the reader who examines Fig. IV.6a below).[8] Though I generally follow Struve, I have often referred to the extensive corrections and comments given by T.E. Peet in his detailed and thoughtful review of Struve's edition.[9]

Problem 1: An aha-problem (? i.e., perhaps one determining an unknown quantity).

Problem 2: A ship's part-problem (i.e., calculating a rudder).

Problem 3: Another ship's part-problem (i.e., calculating a mast).

Problem 4: Calculating the area of a triangle when its meryt (*probably its* altitude) and its teper (base) are given.

Problem 5: A problem concerned with the exchange of 100 loaves of bread for jugs of beer, one involving both their pefsus.

Problem 6: Calculating the sides of a rectangle if its area and the relation of its length to its breadth are given.

Problem 7: Calculating the altitude and base of a triangle when its area and the relation of its altitude to its base are given.

similar to the formula $V = (h/3)(a^2 + a \cdot b + b^2)$ where h is the altitude of the truncated pyramid and a and b are the sides of the bases. The author believed that, if his explanation of the problem was correct, it presented a new and interesting fact, that Egyptian mathematics yielded a problem [and its solution] not yet found in Euclid. We should also mention in passing the early collaboration of Turaev with the mathematician D.P. Tzinserling (also transcribed as Tsinserling), who composed a paper in Russian on "Geometry in Ancient Egypt" that embraced some of Turaev's work on the Moscow Papyrus.[6]

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Problem 7: Calculating the altitude and base of a triangle when its area and the relation of its altitude to its base are given.

Problem 8: A pefsu-problem involving the exchange of bread and beer (with the same numerical data as in Problem 5).

Problem 9: Another pefsu-problem (i.e., a problem involving a given amount of Upper-Egyptian grain to be made partly into bread and partly into beer).

Problem 10: Calculation of the surface of a basket (i.e., of a hemisphere, or, possibly, that of a semi-cylinder; see translation and discussion below).

Problem 11: A baku-problem (i.e., calculating the out-put of a worker using a pehedet-basket).

Problem 12: Another pefsu-problem.

Problem 13: Still another pefsu-problem, the numerical data being the same as in Problem 9.

Problem 14: Calculation of the volume of a frustum of a square pyramid.

Problem 15: An elementary pefsu-calculation.

Problem 16: A pefsu-problem.

Problem 17: Calculating the altitude and base of a triangle when its area and the ratio of its base to its altitude is known.

Problem 18: An area-problem involving the area of a strip of cloth (?).

Problem 19: A simple quantity-problem (i.e., an aha-problem, with the solution indicated).

Problem 20: A pefsu-problem involving loaves of bread and the use of Horus-eye fractions.

Problem 21: A calculation (concerning offering-bread).

Problem 22: A pefsu-problem (i.e., one concerning the production of bread and weak beer from 10 heqat of Upper-Egyptian grain).

Problem 23: A baku-problem (i.e., calculating the output of a shoemaker).

Problem 24: A simple pefsu-problem involving the exchange of bread and beer.

Problem 25: A simple aha-problem with its solution clearly given.

Problem 8: A pefsu-problem involving the exchange of bread and beer (with the same numerical data as in Problem 5).

Problem 9: Another pefsu-problem (i.e., a problem involving a given amount of Upper-Egyptian grain to be made partly into bread and partly into beer).

Problem 10: Calculation of the surface of a basket (i.e., of a hemisphere, or, possibly, that of a semi-cylinder; see translation and discussion below).

Problem 11: A baku-problem (i.e., calculating the out-put of a worker using a pehedet-basket).

Problem 12: Another pefsu-problem.

Problem 13: Still another pefsu-problem, the numerical data being the same as in Problem 9.

Problem 14: Calculation of the volume of a frustum of a square pyramid.

Problem 15: An elementary pefsu-calculation.

Problem 16: A pefsu-problem.

Problem 17: Calculating the altitude and base of a triangle when its area and the ratio of its base to its altitude is known.

Problem 18: An area-problem involving the area of a strip of cloth (?).

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As Struve remarks, the first thing that strikes the reader when he sees the order of the problems presented in the Moscow Papyrus is its complete lack of system. It contrasts strongly with the rather orderly arrangement of the problems in the Rhind Mathematical Papyrus. This leads to the conclusion that the author of the Moscow Papyrus was a student whose training had progressed far enough for the teacher to present various problems to be solved in order to test the skill of the student. The student was apparently not required to present in tabular form, like that found in the Rhind Papyrus, the steps by which the multiplications were carried out, i.e., the doublings, halvings, taking of 2/3 and/or 1/3, and decemplex-multiplications, but rather just to give the results of such multiplications. Still, required or not, such is the abbreviated form in which the solutions are presented in the Moscow tract. However, certain standard forms of terminology and presentation for different kinds of problems are evident in the Moscow Papyrus. Struve in his edition (pp. 12-33) has described the standard phrases in detail. Since I have often added transcriptions as well as translations of them in my document below, where the context and type of problem is evident, I shall not repeat his detailed analysis here. Struve also includes a useful glossary (pp. 187-93).

We should note finally that Struve, in his translation and commentary, does not present the problems in the chaotic order found in the papyrus, but instead he groups together problems of the same type: e.g., [1] problems concerned with ship-parts (pp. 41-44); [2] problems involving bread and beer: their cooking-ratios (pefsu), the exchange of one for the other, the quantities of grain involved in their preparation, and so on (pp. 44-98); [3] sheben-problem, no. 21, closely related to the pefsu problems (pp. 98-101); [4] baku-problems (pp. 101-10); [5] aha-problems (pp. 110-17); [6] volume- and area-problems (pp. 117-69).

I remind the reader that the most important contributions of the Moscow Papyrus are found in the geometrical problems. Among them we can especially note [1] Problem 10, which perhaps concerns the determination of the area of the curved surface of a

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I remind the reader that the most important contributions of the Moscow Papyrus are found in the geometrical problems. Among them we can especially note [1] Problem 10, which perhaps concerns the determination of the area of the curved surface of a hemisphere, as Struve, and later Gillings, believed, but which Peet strongly rejected, or the area of the curved surface of a semi-cylinder, which Peet thought to be more probable, or perhaps indeed some other figure, and [2] Problem 14, the volume of the frustum of a pyramid. These problems and their solutions are discussed below in notes 18 and 24 and above in Chapter Four under the rubric "Volumes." The features of other interesting problems will be noted in the course of my translation.

Finally the reader will see that I have followed the same practices in my translation of this document as in that of the first one. Though I did not have access to any colored photographs of the Moscow Papyrus, I have assumed that Struve was correct in considering the first or title line of each problem as being rubricated (but he underlined the title-lines of the first 16 problems but not those of the last 9). Hence I have followed the procedure of Document IV.1 and given the lines underlined by Struve here in bold-faced type. The words and phrases that are bracketed in my translation indicate that I have added something not specifically in the text in the papyrus. Generally the addition will be clear to the reader from the hieroglyphic transcription given by Struve beneath the hieratic text in Figs. IV.6a-IV.6t, or it will be made clear by the discussion in the endnotes. I have also added in brackets the column and the line numbers for each column of the papyrus. I considered this procedure to be unnecessary in presenting Document IV.1, but here there is so much discussion in the notes of possible other interpretations and readings in Document IV.2 that I thought the numbers would be useful.

Notes to the Introduction to Document IV.2

[1] *Mathematischer Papyrus des Staatlichen Museums der Schöne Künste in Moskau.* (Berlin, 1930), *QSGM.* Abt. A: *Quellen.* Vol. 1 (1930). For a description of the reconstructed papyrus of 45 columns, see pp. 5-7. The true size of the original papyrus is not accurately known because of the existence of the

hemisphere, as Struve, and later Gillings, believed, but which Peet strongly rejected, or the area of the curved surface of a semi-cylinder, which Peet thought to be more probable, or perhaps indeed some other figure, and [2] Problem 14, the volume of the frustum of a pyramid. These problems and their solutions are discussed below in notes 18 and 24 and above in Chapter Four under the rubric "Volumes." The features of other interesting problems will be noted in the course of my translation.

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[2] This letter was included by R.C. Archibald in the supplemental bibliography printed at the end of Vol. 2 of Chace's edition of the Rhind Papyrus, under the year 1930.

[3] See the text over note 1 to the Introduction of Document IV.1.

[4] Struve, *op. cit.* in note 1, pp. 7-12, 38-40.

[5] B.A. Turaeff (or Turejeff or Turaev), "The Volume of the truncated pyramid in Egyptian Mathematics," *Ancient Egypt* (1917), pp. 100-02: "In the collection formerly belonging to Mr. Golenistsheff (!) and recently acquired by the Museum of Fine Arts in Moscow, there is a Mathematical Papyrus of thirty-six columns, in hieratic writing of the epoch of the late Middle Empire. Paleographically it is like some of the Illahun papyri, whilst the breadth of its leaves brings it near to the MSS. of Sinuhe, found in the Ramesseum. This papyrus contains nineteen problems, some of which give us new types of calculation unknown till now, and therefore somewhat difficult to comprehend. Four of these problems are geometrical ones. The first shows how to define the length of the sides of a quadrilateral, when the relation of the sides and the area of the quadrilateral are known. The two next give a method of calculating the area of a triangle: a method already known to us. The fourth presents us, I am inclined to think, with something altogether new in Egyptian scientific literature." In determining the number of problems as 19, he has not considered the fragments; as Struve did when he later showed, by considering them, that 25 problems were originally given. After these opening remarks, Turaev gives a hieroglyphic transcription and an English translation of the fourth geometrical problem, i.e., Problem 14 in Struve's later numbering. Turaev proposes correctly that this problem presents the correct determination of the volume of a frustum of a square pyramid. The author then concludes his paper by saying: "If only our explanation of the problem be right, we have here a new and interesting fact, i.e., the presence in Egyptian mathematics of a problem that is not to be found in Euclid."

[6] D.P. Tsinserling, "Geometriya u drevnikh egiptyan," *Bulletin de l'Académie des Sciences de l'Union des Republiques Soviétiques Socialistes*, Lenigrad, series 6, Vol. 19 (1925), pp. 541-68. Among other subjects, this paper included Turaev's hieroglyphic transcriptions of Problems 1, 2, 12, 15 (old numbering, 6, 7, 17, and 21 in Struve's edition of the Moscow Papyrus.) See Archibald's summary of this paper in Chace's edition of the Rhind Papyrus, Vol. 1, pp. 187-88. See also the mention of Tzinserling by Struve in *op. cit.* in note 1, pp.

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VII-VIII, and particularly the thanks Struve tends to him for their fruitful conversations.

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[8] R.J. Gillings, *Mathematics in the Time of the Pharaohs*, Dover edition (New York, 1982), pp. 246-47, slightly corrected from the original edition (Cambridge, Mass., 1972), has a convenient listing of the problems of the Moscow Papyrus which notes those that are not clear either because they are fragmentary or for some other reason. His list also gives the numerical form of the problem where possible. At the end of the list (p. 247), he makes the following comments [with the numbers in square brackets giving the total number of problems to which each comment refers]:

"On analysis of these problems we find,

"[2 problems] Nos. 1 and 2 are not readable.

"[11] Five problems (8, 9, 13, 22, 24) on the pesus [*Ed.* given in my volume throughout as "pefsu" or "pefsus"] of loaves and beer are not perfectly clear. Three problems (5, 20, 21) deal with the pesu of loaves only. They are difficult to understand. Three problems (12, 15, 16) deal with beer and its pesu only. They are clear and simple.

"[6] Three treat the area of a triangle. No. 4 merely finds the area of a right triangle, while Nos. 7 and 17 are equivalent to the solution of two simultaneous equations, one of the second degree. Two problems (19, 25) concern the solution of equations of the first degree, which are very simple, and No. 6 is on simultaneous equations, one of the second degree.

"[4] Problems 3, 11, 18, and 23 are miscellaneous problems, none of which is entirely clear.

"[2] No. 14 on the volume of a truncated pyramid is a most important problem in the history of Egyptian mathematics. It has no counterpart in any other mathematical papyrus. No. 10 deals, I consider, with the area of the surface of a hemisphere, as Struve thought, and if this is so, it becomes the outstanding Egyptian achievement in the field of mathematics."

[9] *JEA*, Vol. 17 (1931), pp. 154-60.

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DOCUMENT IV.2

The Moscow Mathematical Papyrus

[Problem 1; see Fig. IV.6a, Col. I] [1]
[Lin. 1] ....**it is [subtracted] from what?**
[Lin. 2] ....it is [subtr]acted from what?
[Lin. 3] ....[the resul]t is 5.
[Lin. 4] ....

[Problem 2; see Fig. IV.6a, Col. II] [2]
[Lin. 1] **Example of [the calculation *(irt)*] of a ship's rudder *(hmw)* from..................**
[Lin. 2] If someone says [to you]: "[Take] a ship's rudder [made] from...................
[Lin. 3] "Oh[3] [let me know]....................
[Lin. 4]..

[Problem 3; see Fig. IV.6a, Col. III] [4]
[Lin. 1] **Example of the calculation of a ship's mast from a cedar log.**
[Lin. 2] If someone says to you: "[Make] a mast from a cedar log 30 cubits
[Lin. 3] long [such that the mast] is 1/3 1/5 (?) [of the length of the log]." Calculate 1/3 1/5 of this 30.
[Lin. 4] [If the result is 16, say to him:] "You have obtained
[Lin. 5] [this mast. You have found it] correctly."

[Problem 4; see Fig. IV.6b, Cols. IV-V; see the triangular figure in Col. V with the calculations below it.] [5]
[Lin. 1] **Example of the calculation of [the area of] a triangle.**

[Lin. 2] [If someone says to you:] "[Assume] a triangle of 10 [khet] on the [*mr*]*yt* (i.e., most likely the 'altitude' or 'kathete')
[Lin. 3] and 4 [khet] on the base. [Make] known to me
[Lin. 4] [its area." Take 1/2 of] this [4,]
[Lin. 5] [namely 2, in order to get one side of its equivalent rectangle. Multiply 10, the other side of the rectangle,] times 2.
[Lin. 6] [It becomes 20. Its area] is this.

[Problem 5; see Fig. IV.6b, Cols. VI-VII]
[Col. VI]
[Lin. 1] **Example of calculating 100 loaves of bread of pefsu 20.**
[Lin. 2] If someone says t[o you]: "[You have] 100 loaves of bread of pefsu 20,
[Lin. 3] [they] have been exchanged[6] for beer of pefsu 4,
[Lin. 4] [the beer being like the 1/2 1/4 kind of beer[7]], [How is it to be calculated?]" [First] calculate the quantity of flour] needed
[Lin. 5] [for these 100 loaves of bread] of pefsu 20. The result is 5 heqat.
[Lin. 6] [For this weaker beer that is like 1/2 1/4 beer, you seek] what you need for 1 [des-jug].
[Lin. 7] The result is 1/2 [of what you need for a 1 des-jug of beer made from Upper-Egyptian grain].

[Col. VII]
[Lin. 1] Calculate
[Lin. 2] 1/2 of 5 heqat. The result is
[Lin. 3] 2 1/2 heqat. Multiply
[Lin. 4] 2 1/2 times 4.
[Lin. 5] The result is 10 [des-jugs].
[Lin. 6] Behold this is its beer quantity.
[Lin. 7] You will find it [to be correct].

[Problem 6; see Fig. IV.6c, Col. VIII][8]
[Lin. 1] **Example of calculating a rectangle.**

[Lin. 2] [If someone says to you:] "[Assume] a triangle of 10 [khet] on the [*mr*]*yt* (i.e., most likely the 'altitude' or 'kathete')
[Lin. 3] and 4 [khet] on the base. [Make] known to me
[Lin. 4] [its area." Take 1/2 of] this [4,]
[Lin. 5] [namely 2, in order to get one side of its equivalent rectangle. Multiply 10, the other side of the rectangle,] times 2.
[Lin. 6] [It becomes 20. Its area] is this.

[Problem 5; see Fig. IV.6b, Cols. VI-VII]
[Col. VI]
[Lin. 1] **Example of calculating 100 loaves of bread of pefsu 20.**
[Lin. 2] If someone says t[o you]: "[You have] 100 loaves of bread of pefsu 20,
[Lin. 3] [they] have been exchanged[6] for beer of pefsu 4,
[Lin. 4] [the beer being like the 1/2 1/4 kind of beer[7]], [How is it to be calculated?]" [First] calculate the quantity of flour] needed
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[Lin. 7] The result is 1/2 [of what you need for a 1 des-jug of beer made from Upper-Egyptian grain].

[Col. VII]
[Lin. 1] Calculate
[Lin. 2] 1/2 of 5 heqat. The result is
[Lin. 3] 2 1/2 heqat. Multiply
[Lin. 4] 2 1/2 times 4.
[Lin. 5] The result is 10 [des-jugs].
[Lin. 6] Behold this is its beer quantity.
[Lin. 7] You will find it [to be correct].

[Problem 6; see Fig. IV.6c, Col. VIII][8]
[Lin. 1] **Example of calculating a rectangle.**

[Lin. 2] If someone says to you: "A rectangle 12 setjat[9] [in area] [has] a breadth 1/2 1/4 of its length. [Calculate its area.]"
[Lin. 3] Calculate 1/2 1/4 to get 1. The result is 1 1/3.
[Lin. 4] Take this 12 setjat 1 1/3 times. The result is 16.
[Lin. 5] Calculate its square root.[10] The result is 4 for its length; [and] 1/2 1/4 of it is 3 for
[Lin. 6] the breadth.[11] The correct procedure is as follows [see the rectangle in line 6 of column VIII in Fig. IV.6c, marked with the area of 12 in the center, the length of 4 above, and the breadth of 3 on the left side. The figure illustrates the problem and represents a kind of proof. Then follows the calculation of the area, which shows that 3 x 4 does indeed equal 12, the specified area]:

\ 1	4
\ 2	8
[Total:	12].

[Problem 7; see Fig. IV.6c, Col. IX][12]
[Lin. 1] **Example of calculating a triangle.**
[Lin. 2] If someone says to you: "[There is] a triangle[13] with area of 20 [setjat] and 'bank' (*ldb*, i.e., the ratio of height to base) of 2 1/2."
[Lin. 3] Double the area. The result is 40. Take it 2 1/2 times.
[Lin. 4] The result is 100. Take the square root; the result is 10. Call up 1 from 2 1/2.
[Lin. 5] The result is 1/3 1/15. Apply this to 10. The result is 4.
[Lin. 6] [Hence it is] 10 [khet] in the length (i.e., kathete) and 4 [khet] in its breadth.[14]

[Problem 8; see Fig. IV.6d, Cols. X-XI][15]
[Col. X]
[Lin. 1] **Example of calculating 100 loaves of bread of pefsu 20.**
[Lin. 2] If someone says to you: "[You have] 100 loaves of bread of [pefsu] 20
[Lin. 3] to be exchanged[16] for beer of pefsu 4
[Lin. 4] like 1/2 1/4 malt-date beer."

[Lin. 2] If someone says to you: "A rectangle 12 setjat[9] [in area] [has] a breadth 1/2 1/4 of its length. [Calculate its area.]"
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[Total:	12].

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[Lin. 1] **Example of calculating a triangle.**
[Lin. 2] If someone says to you: "[There is] a triangle[13] with area of 20 [setjat] and 'bank' (*ldb*, i.e., the ratio of height to base) of 2 1/2."
[Lin. 3] Double the area. The result is 40. Take it 2 1/2 times.
[Lin. 4] The result is 100. Take the square root; the result is 10. Call up 1 from 2 1/2.
[Lin. 5] The result is 1/3 1/15. Apply this to 10. The result is 4.
[Lin. 6] [Hence it is] 10 [khet] in the length (i.e., kathete) and 4 [khet] in its breadth.[14]

[Problem 8; see Fig. IV.6d, Cols. X-XI][15]
[Col. X]
[Lin. 1] **Example of calculating 100 loaves of bread of pefsu 20.**
[Lin. 2] If someone says to you: "[You have] 100 loaves of bread of [pefsu] 20
[Lin. 3] to be exchanged[16] for beer of pefsu 4
[Lin. 4] like 1/2 1/4 malt-date beer."

[Lin. 5] [First] calculate the [grain] required for the 100 loaves of bread of pefsu 20.
[Lin. 6] The result is 5 [heqat]. [Then] reckon what you need for a 1 des-jug of the beer [like] the beer [called] 1/2 1/4 malt-date beer.
[Lin. 7] The result is 1/2 [of the heqat-measure needed for 1 des-jug of beer made from Upper-Egyptian grain].

[Col. XI]
[Lin. 1] Calculate 1/2 of 5 [heqat]. The result will be 2 1/2.
[Lin. 2] Take this 2 1/2 four times.
[Lin. 3] The result is 10. Then you say to him:
[Lin. 4] "Behold! Its beer quantity is found to be correct."

[Problem 9; see Figs. IV.6e-f, Cols. XII-XVII]
[Col. XII]
[Lin. 1] **Calculating [16] heqat of Upper-Egyptian grain for bread and for beer.** [17]
[Lin. 2] If someone says to you: "[You have] 16 heqat of Upper-Egyptian grain; calculate the amount for 100 loaves of bread of pefsu 20,
[Lin. 3] leaving the rest for beer of 2 pefsu,
.[Lin. 4] of 4 pefsu,
[Lin. 5] and of 6 pefsu
[Lin. 6] [like] 1/2, 1/4 malt-date beer."

[Col. XIII]
[Lin. 1] [First] Reckon the required amount of grain for the 100 loaves of bread of pefsu 20.
[Lin. 2] It is 5 heqat of Upper-Egyptian grain. Calculate the remainder
[Lin. 3] from the 16 heqat of Upper-Egyptian grain. The result is 11 heqat of Upper-Egyptian grain.
[Lin. 4] You say to him: "The 11 heqat of Upper-Egyptian grain is what is turned into

[Lin. 5] [First] calculate the [grain] required for the 100 loaves of bread of pefsu 20.
[Lin. 6] The result is 5 [heqat]. [Then] reckon what you need for a 1 des-jug of the beer [like] the beer [called] 1/2 1/4 malt-date beer.
[Lin. 7] The result is 1/2 [of the heqat-measure needed for 1 des-jug of beer made from Upper-Egyptian grain].

[Col. XI]
[Lin. 1] Calculate 1/2 of 5 [heqat]. The result will be 2 1/2.
[Lin. 2] Take this 2 1/2 four times.
[Lin. 3] The result is 10. Then you say to him:
[Lin. 4] "Behold! Its beer quantity is found to be correct."

[Problem 9; see Figs. IV.6e-f, Cols. XII-XVII]
[Col. XII]
[Lin. 1] **Calculating [16] heqat of Upper-Egyptian grain for bread and for beer.** [17]
[Lin. 2] If someone says to you: "[You have] 16 heqat of Upper-Egyptian grain; calculate the amount for 100 loaves of bread of pefsu 20,
[Lin. 3] leaving the rest for beer of 2 pefsu,
.[Lin. 4] of 4 pefsu,
[Lin. 5] and of 6 pefsu
[Lin. 6] [like] 1/2, 1/4 malt-date beer."

[Col. XIII]
[Lin. 1] [First] Reckon the required amount of grain for the 100 loaves of bread of pefsu 20.
[Lin. 2] It is 5 heqat of Upper-Egyptian grain. Calculate the remainder
[Lin. 3] from the 16 heqat of Upper-Egyptian grain. The result is 11 heqat of Upper-Egyptian grain.
[Lin. 4] You say to him: "The 11 heqat of Upper-Egyptian grain is what is turned into

[Col. XIV
[Lin. 1] beer of pefsu 2,
[Lin. 2] of pefsu 4,
[Lin. 3] of pefsu 6,
[Lin. 4] like 1/2 1/4 malt-
[Lin. 5] date beer."

[Col. XV]
[Lin. 1] [First] reckon the grain required [for 1 des-jug of the beer of] pefsu 2. The result is 1/2.
[Lin. 2] Reckon the grain required [for 1 des-jug of the beer of] pefsu 4. The result is 1/4.
[Lin. 3] Reckon the grain required [for 1 des-jug of the beer of] pefsu 6. The result is 1/6.
[Lin. 4] Add them together. The result is 2/3 1/4.
[Lin. 5] Take 2/3 1/4 two times, because it was said to him,

[Col. XVI]
[Lin. 1] [it is like] 1/2 1/4 malt-
[Lin. 2] date beer.
[Lin. 3] The result is 1 2/3 1/6.
[Lin. 4] Calculate with 1 2/3 1/6
[Lin. 5] to find 11,

[Col. XVII]
[Lin. 1] which resulted as the remainder from those 16 heqat of the Upper-Egyptian grain after these 5 heqat of the Upper-Egyptian grain [were used for bread].
[Lin. 2] The result is 6. Say to him. "Behold! This is what has been brought (!, but actually, what can be produced) of each [beer of varying] pefsu, [namely,]
[Lin. 3] 6 1-des-jugs of beer of each pefsu.
[Lin. 4] You know it. You have what it is.
[Lin. 5] Proceeding in the manner that has been developed, you will find [it to be correct]."

[Col. XIV
[Lin. 1] beer of pefsu 2,
[Lin. 2] of pefsu 4,
[Lin. 3] of pefsu 6,
[Lin. 4] like 1/2 1/4 malt-
[Lin. 5] date beer."

[Col. XV]
[Lin. 1] [First] reckon the grain required [for 1 des-jug of the beer of] pefsu 2. The result is 1/2.
[Lin. 2] Reckon the grain required [for 1 des-jug of the beer of] pefsu 4. The result is 1/4.
[Lin. 3] Reckon the grain required [for 1 des-jug of the beer of] pefsu 6. The result is 1/6.
[Lin. 4] Add them together. The result is 2/3 1/4.
[Lin. 5] Take 2/3 1/4 two times, because it was said to him,

[Col. XVI]
[Lin. 1] [it is like] 1/2 1/4 malt-
[Lin. 2] date beer.
[Lin. 3] The result is 1 2/3 1/6.
[Lin. 4] Calculate with 1 2/3 1/6
[Lin. 5] to find 11,

[Col. XVII]
[Lin. 1] which resulted as the remainder from those 16 heqat of the Upper-Egyptian grain after these 5 heqat of the Upper-Egyptian grain [were used for bread].
[Lin. 2] The result is 6. Say to him. "Behold! This is what has been brought (!, but actually, what can be produced) of each [beer of varying] pefsu, [namely,]
[Lin. 3] 6 1-des-jugs of beer of each pefsu.
[Lin. 4] You know it. You have what it is.
[Lin. 5] Proceeding in the manner that has been developed, you will find [it to be correct]."

[Problem 10 following the text and to some extent the German translation of Struve; see Fig. IV.6g, Cols. XVIII-XX; cf. Fig. IV.7]
[Col. XVIII][18]

[Lin. 1] **Example of calculating a basket** (⌓ ⌣ |, *nbt*) [assumed by Struve as hemispheric in shape; see Fig. IV.8]
[Lin. 2] If someone says to you: "A basket with a mouth opening
[Lin. 3] of 4 1/2 (i.e., a diameter of this size) in good condition (*ʿd*), oh
[Lin. 4] let me know its [surface] area (*ꜣḥt*)."
[Lin. 5] [First] calculate 1/9 of 9, since the basket is
[Lin. 6] 1/2 of an egg-shell (*? inr?*). The result is 1.

[Col. XIX]
[Lin. 1] Calculate the remainder as 8.
[Lin. 2] Calculate 1/9 of 8.
[Lin. 3] The result is 2/3 1/6 1/18. Cal-
[Lin. 4] culate the remainder from these 8 after
[Lin. 5] taking away those 2/3 1/6 1/18. The result is 7 1/9.

[Col. XX]
[Lin. 1] Reckon with 7 1/9 four and one-half times.
[Lin. 2] The result is 32. Behold, this is its area.
[Lin. 3] You will find that it is correct.

[An interpretation of Problem 10 as concerned with the area of the curved surface of a half-cylinder outlined by T. E. Peet, "A Problem in Egyptian Geometry," *JEA*, Vol. 17 (1931), pp. 104-06, and p. 105. Peet included the following translation, with the lines numbered consecutively rather than by each of the papyrus columns:[19]]

1. *Example of working out a semi-cylinder.*
2. *If they say to you, A semi-cylinder <of 4 1/2> in diameter*
3. *by 4 1/2 in height; pray*

[Problem 10 following the text and to some extent the German translation of Struve; see Fig. IV.6g, Cols. XVIII-XX; cf. Fig. IV.7]
[Col. XVIII][18]

[Lin. 1] **Example of calculating a basket** (⌓ ǀ, *nbt*) [assumed by Struve as hemispheric in shape; see Fig. IV.8]
[Lin. 2] If someone says to you: "A basket with a mouth opening
[Lin. 3] of 4 1/2 (i.e., a diameter of this size) in good condition (*ʿḏ*), oh
[Lin. 4] let me know its [surface] area (*ȝḥt*)."
[Lin. 5] [First] calculate 1/9 of 9, since the basket is
[Lin. 6] 1/2 of an egg-shell (*? lnr?*). The result is 1.

[Col. XIX]
[Lin. 1] Calculate the remainder as 8.
[Lin. 2] Calculate 1/9 of 8.
[Lin. 3] The result is 2/3 1/6 1/18. Cal-
[Lin. 4] culate the remainder from these 8 after
[Lin. 5] taking away those 2/3 1/6 1/18. The result is 7 1/9.

[Col. XX]
[Lin. 1] Reckon with 7 1/9 four and one-half times.
[Lin. 2] The result is 32. Behold, this is its area.
[Lin. 3] You will find that it is correct.

[An interpretation of Problem 10 as concerned with the area of the curved surface of a half-cylinder outlined by T. E. Peet, "A Problem in Egyptian Geometry," *JEA*, Vol. 17 (1931), pp. 104-06, and p. 105. Peet included the following translation, with the lines numbered consecutively rather than by each of the papyrus columns:[19]]

1. *Example of working out a semi-cylinder.*
2. *If they say to you, A semi-cylinder <of 4 1/2> in diameter*
3. *by 4 1/2 in height; pray*

4. *let me know its area. You are to*
5. *take a ninth of 9, since a semi-cylinder*
6. *is half of a [cylinder]; result 1.*
7. *Take the remainder, namely 8.*
8. *You are to take a ninth of 8;*
9. *result 2/3 + 1/6 + 1/18. You are to take*
10. *the remainder of the 8 after (subtraction of)*
11. *the 2/3 + 1/6 + 1/18, result 7 1/9.*
12. *You are to take 7 1/9 4 1/2 times;*
13. *result 32. See, this is its area.*
14. *You will find it correct.*

[Problem 11; see Fig. IV.6h,Cols. XXI-XXII][20]
[Col. XXI]
[Lin. 1] **Example of reckoning the work of a man in logs.**
[Lin. 2] If someone says to you: "The work of a man in logs;
[Lin. 3] the amount of his work is 100 logs
[Lin. 4] of 5 handbreadths section; but he has brought them in logs
[Lin. 5] of 4 handbreadths section." You are to square these 5 handbreadths. The result is
[Lin. 6] 25. You are to square the 4 handbreadths. The result is 16.

[Col. XXII]
[Lin. 1] Reckon with this 16 to get 25.
[Lin. 2] The result is 1+1/2+1/16 times. You are to take this number 100 times.
[Lin. 3] The result is 156 1/4 [corr. ex 1/2 1/16 in papyrus!]. Then you shall say to him, "Behold,
[Lin. 4] this is the number of logs which he brought of 4 handbreadths section.
[Lin. 5] You will find that it is correct."

[Problem 12; see Fig. IV.6i, Cols. XXIII-XXIV]
[Col. XXIII] *[cont.]*

[Lin. 1] **Example of the calculation of 13 heqat of Upper-Egyptian grain.**
[Lin. 2] If someone says to you: "[Take] 13 heqat (*lit.* 1 ten-heqat and 3 heqat) of Upper-Egyptian grain to make [them]
[Lin. 3] into 18 1-des jugs of beer [like] malt-
[Lin. 4] date [beer]," See (i.e., note) that
[Lin. 5] the amount of grain for 1-des of Upper-Egyptian beer [like] malt-date beer is
[Lin. 6] 2 1/6. Reckon with 2 1/6 in order
[Lin. 7] to find 13, for note that 13 [as a simple number]
[Lin. 8] is the [same as] the 1 ten-heqat plus 3 heqat [given above in parentheses].[21] The result is 6 times.

[Col. XXIV]
[Lin. 1] Reckon with 6 to find 18.
[Lin. 2] The result is 3 times. Behold, [this is] the pefsu,
[Lin. 3] [namely] 3. You will find [that] it is correct.[22]

[Problem 13; see Figs. IV.6i and IV.6j, Cols. XXIV-XXVI.][23]
[Col. XXIV]
[Lin. 4] **Calculating 16 heqat of Upper-Egyptian grain,**
[Lin. 5] reckoning it as 100 loaves of bread of pefsu 20, [leaving] the rest
[Lin. 6] for beer of 2 pefsu,
[Lin. 7] of 4 pefsu,
[Lin. 8] and of 6 pefsu

[Col. XXV]
[Lin. 1] [like] 1/2, 1/4 malt-date beer.
[Lin. 2] Reckon the required amount of grain for the 100 loaves of pefsu 20.
[Lin. 3] The result is 5. Calculate the remainder from the 16
[Lin. 4] after the 5 [have been subtracted]. The result is 11. Divide
[Lin. 5] it up among each of the three different pefsus.

[Lin. 6] The result is 2/3 1/4. Take 2/3 1/4 two times

[Col. XXVI]
[Lin. 1] because it was said to him, "[it is like] 1/2 1/4 malt-date beer."
[Lin. 2] The result is 1 2/3 1/6. Calculate with this 1 2/3 1/6 to
[Lin. 3] find 11. The result is 12 (!, *should be* 6) times.
[Lin. 4] Say to him, "This is your beer. You will find that it is correct."

[Problem 14; see Fig. IV.6j-IV.6k, Cols. XXVII-XXIX]
[Col. XXVII][24]
[Lin. 1] **Example of calculating a truncated [square] pyramid.**
[Lin. 2] If someone says to you: "A pyramid of 6 for the height (*šṯwtī*)
[Lin. 3] by 4 on the base (i.e., the side of the lower square) by 2 on the top (i.e., the side of the upper square)."
[Lin. 4] You are to square this 4; the result is 16.
[Lin. 5] You are to double 4 (i.e., multiply 4 by 2); the result is 8.
[Lin. 6] You are to square this 2; the result is 4.

[Col. XXVIII]
[Lin. 1] You are to add the 16
[Lin. 2] and the 8 and the 4;
[Lin. 3] the result is 28. You are to take
[Lin. 4] 1/3 of 6; the result is 2. You are to take 28 two times; the result is 56.
[Lin. 5] Behold, [the volume] is 56. You will find [that this is] correct.

[Col. XXIX]
[For the diagram given in this column with translated numerals and their computation, see Fig. IV.10.]

[Problem 15; see Fig. IV.6k, Col. XXX][25]

[Lin. 1] **The calculation of 10 heqat of Upper-Egyptian Grain.**
[Lin. 2] If someone says to you: "10 heqat of Upper-Egyptian Grain;
[Lin. 3] [they] are to be made into beer of pefsu 2.
[Lin. 4] Oh let me know [the amount of]
[Lin. 5] the beer." Reckon with this 10
[Lin. 6] two times. The result is 20. Behold,
[Lin. 7] the beer [quantity] is 20 1-des-jugs. You will find it right.

[Problem 16; see Fig. IV. 6L, Col. XXXI-XXXII][26]
[Col. XXXI]
[Lin. 1] **Example of the calculation of des-jugs of beer of pefsu 2.**
[Lin. 2] If someone says to you: "Des-jugs of beer of pefsu 2
[Lin. 3] [like] 1/2 1/4 malt-beer,
[Lin. 4] a quantity of three 1-des-jug [made from] 3 heqat with the 2 2/3 measure." Calculate
[Lin. 5] the [grain] required for a 1-des-jug of this beer with the pefsu of 2.
[Lin. 6] The result is 1/2 [heqat]. Take it 2 times.
[Lin. 7] The result is 1. Reckon with 2 2/3 to find 1.

[Col. XXXII]
[Lin. 1] The result is 1/4 1/8. Calculate
[Lin. 2] 1/4 1/8 of 1/3. The result is 1/8 [of a heqat]. Do
[Lin. 3] say to him: "This is it.
[Lin. 4] You will find that it is correct."

[Problem 17; see Fig. IV.6m, Cols. XXXIII-XXXIV]
[Col. XXXIII][27]
[Lin. 1] Example of calculating a triangle.
[Lin. 2] If someone says to you: "A triangle of 20 [setjat] in its area (*ꜣḥt*)
[Lin. 3] and what you put on the length, you must put 1/3 1/15 (i.e., 2/5) thereof on its breadth."

[Lin. 4] Double the 20 [setjat]; the result is 40.
[Lin. 5] Reckon with 1/3 1/5 so as to find 1. The result is 2 1/2 times.
[Lin. 6] Reckon with 40 2 1/2 times. The result is 100. You are to take its square root.

[Col. XXXIV]
[Lin. 1] The result is 10. Behold it is 10 [khet] in length. You are to take 1/3 1/15.
[Lin. 2] of 10. The result is 4. Behold, it is 4 [khet] on the breadth. You will find [it to be] correct.
[The remainder of Col. XXXIV is the diagram of the triangle and its measurements, plus the calculations already specified. In addition to the figure as presented in Fig. IV.6m, see also Fig. IV.11 presented with the numerical data translated.]

[Problem 18; see Fig. IV.6n, Col. XXXV]
[The rendition given by Struve is, as Peet points out,[28] an unsatisfactory reconstruction, but I give it here in part (with substantial changes in the last two lines) for want of a more satisfactory version.]
[Lin. 1] Example of the calculation of a strip of garment-cloth 5 cubits [plus] 5 palms[29] [long by] 2 palms [wide], the area of which is to be reckoned.
[Lin. 2] If someone says to you, "A strip of garment-cloth 5 cubits 5 palms by 2 palms, the area of which is to be reckoned,
[Lin. 3] Oh let me know its area." Convert
[Lin. 4] this strip of 5 cubits 5 palms [by] 2 palms into palms [where 1 cubit = 7 palms]. The result is 35 [palms in length] for this 5 cubits
[Lin. 5] [while] the result for [the area of the end piece of] this strip is 10 (i.e., 5 x 2). [Then you] multiply 35 times [2, the width], and add [this 70 to the area] 10 [found for the end piece], the [resulting total area of the cloth strip is] 80 [square palms].

[Problem 19; see Fig. IV.6n, Col. XXXVI][30]

[Lin. 1] Example of calculating a quantity taken 1 and 1/2 times and added to
[Lin. 2] 4 to make 10. What is the quantity which says this (i.e., that produces this equality)?
[Lin. 3] Calculate the excess of this 10 over 4. The result is 6.
[Lin. 4] You operate on 1 1/2 to find 1. The result is 2/3. You
[Lin. 5] take 2/3 of this 6. The result is 4. Behold, 4
[Lin. 6] says it (i.e., satisfies the equality given above). You will find [that this is] correct.

[Problem 20; see Fig. IV.6o, Col. XXXVII]

[Lin. 1] Example of the calculation of 1000 loaves of bread of pefsu 20.
[Lin. 2] If someone says to you: "1000 loaves of bread of pefsu 20, like that which has come from and is filled with emmer [alone];
[Lin. 3] Make known to me the emmer." Reckon with 20 to find 2 2/3.
[Lin. 4] The result is 1/5 of 2/3. Take 1/5 of 2/3 of this 1000. The result is 133 1/3.
[Lin. 5] [If] you calculate it in Upper-Egyptian grain, the result is 133 *1/4* + *1/16* + *1/64* heqat 1 2/3 ro.[31]

[Problem 21; see Fig. IV.6p, Cols. XXXVIII-XXXIX][32]

[Col. XXXVIII]
[Lin. 1] Example of calculating the mixing (*šʿbn* = *šbn*) of offering-bread.
[Lin. 2] If someone says to you: "20 measured (?) [as Horus-eye fraction] (i.e., *1/8* heqat of grain) and 40 measured (?) as [Horus-eye fraction] (i.e., *1/16* heqat of grain)."
[Lin. 3] You are to take 1/8 of 20; because the [Horus-eye sign] is 1/8.
[Lin. 4] The result is 2 1/2. You are to take 1/16 of 40 because

[Lin. 5] [the Horus-eye sign]  is 1/16. The result is 2 1/2. You are to calculate
[Lin. 6] the total of these [fractions of 20 and 40]. The result is 5. You are to calculate the total

[Col. XXXIX]
[Lin. 1] of these [initial numbers 20 and 40]. The result is 60. Then you divide 5 by 60, and
[Lin. 2] the result is 1/12 (*corr. ex* 1/16). Behold, the mixture is 1/12 (*corr. ex* 1/16). You will find [that it is] correct.

[Problem 22; see Fig. IV.6q, Cols. XL-XLI]
[Col. XL]
[Lin. 1] Example of calculating with 10 heqat of Upper-Egyptian grain.
[Lin. 2] If someone says to you: "10 heqat of Upper-Egyptian grain
[Lin. 3] are to be calculated like 100 loaves of bread whose pefsu is not known.
[Lin. 4] and the remainder is for 10 [des-jugs of] beer with pefsu 2, like 1/2 1/4 malt-date [beer]."
[Lin. 5] Behold, 1/2 1/4 malt-date beer.
[Lin. 6] Behold it is 2 des-jugs [of beer prepared from grain]. You will calculate the portion of the 10 des-jugs of beer with [pefsu] 2.
[Lin. 7] The result is [5 heqat]. [You will] calculate the remainder of this 10 [des-jugs of beer] in accordance with this [5 heqat] of Upper-Egyptian grain.

[Col. XLI]
[Lin. 1] The result is 5. You will calculate [as] with 1/2 1/4 malt-date [beer] in order to find 1.
[Lin. 2] Behold, [as] with 1/2 1/4 malt-date [beer you will get] 2 [jugs for 1 of stronger beer?]. The result is [that you need] 1/2 [the grain needed for the first portion of producing stronger beer]. [So] you will take 1/2 of 5.

[Lin. 3] The result is 2 *1/2* [heqat].[33]

[Problem 23; see Fig. IV.6r, Col. XLII and particularly the corrections and interpretation suggested by Peet in his review (pp. 158-59)][34]

[Lin. 1] Example of reckoning the work of a shoemaker.
[Lin. 2] If someone says to you: "[Regarding] the work of a shoemaker, if he is cutting out [only], [he can do] 10
[Lin. 3] [pairs of sandals] per day; [but] if he is decorating, [he can do] 5 per day.
[Lin. 4] As for [the number] he can both cut and decorate in a day,
[Lin. 5] what will that be?" You will calculate [the sum of] the [day-]equivalencies (*rmnʿwy* or *rmnwy*) of the 10 and the 5 (i.e., add the 1 day for cutting out the 10 pairs of sandals and the 2 days for decorating them).
[Lin. 6] The result for them together is 3 [days]. Take this to find 10. The result is 3 1/3 times. Behold it is 3 1/3 [pairs of sandals] per day [to be fully cut and decorated].

[Problem 24; see Fig. IV.6s, Cols. XLIII-XLIV][35]

[Col. XLIII]
[Lin. 1] Example of calculating with 15 heqat of Upper-Egyptian grain.
[Lin. 2] If someone says to you, "[There are] 15 heqat of Upper-Egyptian grain to be made into 200 loaves of bread.
[Lin. 3] The remainder [is to be made into] 10 des-jugs of beer of pefsu 1/10 that of the bread, the pefsu of the beer being [like]
[Lin. 4] 1/2 1/4 malt-date beer." Reckon with 1/10 to find 1.
[Lin. 5] The result is 10 times. Reckon with the 10 des-jugs of beer
[Lin. 6] 10 times. The result is 100. Add the 100 to the 200.
[Lin. 7] The result is 300. Reckon with 15 to find 300. The result is 20 times.

[Col. XLIV]

[Lin. 1] Behold, [this] 20 is the pefsu of the 100 (!, *should be* 200) loaves of bread.
[Lin. 2] Take 1/10 of the 20. The result is 2.
[Lin. 3] Behold what belongs to the 10 des-jugs of beer; see, it is the pefsu 2
[Lin. 4] You will find [it to be] correct.

[Problem 25; see Fig. IV.6t, Col. XLV]
[Lin. 1] Example of calculating a quantity such that if it is taken two times along with [the quantity itself], it (i.e., the sum) comes to 9.
[Lin. 2] What is the quantity that says it (i.e., satisfies the statement).[36] Reckon the sum of this quantity [taken as 1] and the 2.
[Lin. 3] The result is 3. Reckon with 3 to find 9. The result is 3 times.
[Lin. 4] Behold 3 says it (i.e., satisfies the statement). You will find [that it is] correct.

Notes to Document IV.2

[1] As I have said before, Struve's reconstruction is completely speculative. The reader will see that this is so if he examines Fig. IV.6a, Problem 1. The bracketed words in the first line come from the fact that apparently the verb *pri* also appeared in the second line and it is conventional in this text to repeat the title line in stating the problem in the second line. We can give some credence to Struve's belief that this is an aha-problem similar to Problem 28 in the Rhind Papyrus where *pri* is used to mean "take away" or "subtract" (see my Fig. IV.2z). He sees in a fragment the "20" which appears to be the remainder when 1/5 of the unknown quantity is "subtracted" from the unknown quantity. Apparently in the third line the author said something like "operate on 1/5 to get 1" since the fragment of the third line seems to tell ns that the result is 5. Subtracting 1 from 5 we get 4, i.e., 4/5 of 1. But the remainder in fact was 20, so that we must find what multiplier makes 4/5 into 20. The answer is 25; hence 25 is the desired "aha," i.e., unknown quantity. Thus, as in the Rhind Papyrus, we perhaps have here an arithmetic problem solved like an algebraic solution of a linear equation by means of the initial assumption of the "aha" as "1," i.e., by false position.

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[2] It is not clear what is being calculated in this problem because of the fragmentary nature of the text. It probably has something to do with the relationship of rudder length to the ship's length or size.

[3] See Peet's review in JEA, Vol. 17 (1931), p. 154: "P. 42, n. 2 [cf. my Fig. IV.6a]. The reading in ii, 3 can hardly be right. The first sign is surely , and, though the second with its small head, and its lower stroke almost at the level of the top of the *ḥ*, is not a convincing , I am inclined to think that is what it is."

[4] I have followed Struve's reconstruction (*op. cit.*, p. 43) for the most part. If this is a correct reconstruction, then we have a very simple problem involving fractional multiplication. But Peet in his review, *ibid.*, p. 154, has a different reading of some of the signs that casts doubt on Struve's reconstruction: "P. 43, no. 3. The interpretation and restoration given [by Struve] are unsatisfactory because *ḥt-tꜣw n ꜥš* cannot mean 'a mast (made) out of a cedar' but only a 'cedar mast,' and because line 3 as restored could not possibly convey the meaning required of it, namely that the mast should be 1/3 + 1/5 of the height of the cedar. The readings here given are not all correct. In the first place S. has failed to notice that the small square projection of papyrus at the bottom of the left-hand piece (Fragment 2) has been wrongly mounted. It should be swung round through a right angle to the left on its left-hand top corner. It then completes the *n* of *pn* in l. 3 and the word *ꜥšpt* (apparently so) in line 4. The signs under the *p* are, in the tattered state of the papyrus, not certain; is impossible. In l. 3 after the traces of *ꜣw* 1/3 S. reads (p. 43, fig.2) the fraction 1/5. No fraction stood here; what remains might be a trace of , and there may be room for a horizontal sign below it. The sign which precedes in l. 3 might, as S. thinks, be 1/3 (is not possible), but in this case what is the dot to the left of its top, and why a stroke after it? The sign transcribed by S. disappears when the loose fragment is correctly placed.

"I have no constructive criticism to make on this problem. It is possible that it dealt with the volume of a mast 30 cubits long and so many handbreadths in diameter of section."

[5] As Struve (*op. cit.*, p. 146) notes, this problem is equivalent to Problem 51 in the Rhind Papyrus, and so see my discussion of that problem in Chapter Four in the section on areas and also in Document IV.1, note 68. Because of its similarity to Problem 51 of Document IV.1, I have suggested additions to the in-

complete lines 4-6 in the Moscow Papyrus. Note in Col. V (lines 1 and 2 of my Fig. IV.6b) the incomplete figure of a triangle (either an isosceles or a right triangle) with the lengths of the *mryt* (10) and base (4), plus the total area (20, written as two strokes). Under the figure are the following calculations in lines 3 and 4:

1	4	1	[10]
1/2	2	\ 2	[20].

Compare the calculations of Problem 51 in Document IV.1. I have already discussed in connection with that problem (n. 68) Gunn and Peet's view of the *mryt* as the perpendicular height or kathete of a triangle.

[6] Peet argues, *op. cit.*, p. 157 that the word here and in other bread to beer problems should be (*db3*) and not (*ʿpr*), i.e., "exhange" not "provide."

[7] This is apparently a shortened form of the so-called "1/2 1/4 malt-date beer" given below in line 4 of Col. X in Problem 8, which, except for this use of the fuller form of the name of the weak beer, is identical with Problem 5.

For a brief discussion of pefsu, see Chapter IV, under the rubric "Pefsu Problems," and endnotes 103-04 of Document IV.1, where it is mentioned that the pefsu (cooking ratio, i.e., the ratio of the quantity of bread or beer to the quantity of meal used in the cooking) of bread or beer indicates inversely the strength of the bread or beer after it has been cooked or brewed.

Note also that there is a long discussion of pefsu problems in Struve's edition, pp. 44-101. Struve always uses the spelling pesu, as does Gillings. In my translatious I have always used pefsu for the sake of consistency. Struve's treatment of the kind of weak beer mentioned in this and other problems in the Moscow Papyrus and his attempt to untangle the role that the name of the weak beer (i.e., 1/2 1/4 malt-date beer) plays in the calculation of this and other problems seem puzzling to me. Rather, it appears to me that the fractions that are a part of the name of the beer do not play the role that Struve assigns to them in the calculations of Problems 5 and 8; but rather, as the pefsu of 4 of the beer in this problem suggests, the name of the weak beer only serves to indicate that this is a weaker beer like that of beer made from a mixture of malt and dates whose components are somehow represented by the fractions 1/2 and 1/4. The usual pefsu of stronger beer is 2. Hence the beer of pefsu 4 needs only half the quantity of that of pefsu 2; hence the conclusion in Column VI, line 7, that the "result is 1/2." Therefore, in determining the jugs of beer here, the author takes 1/2 the heqat measure of meal that produced the 100 loaves of bread with pefsu 20 in order to produce 10 jugs of beer of pefsu 4.

I prefer the discussion of this problem given by Peet in his review, pp. 155-56, but it too seems inconclusive to me. It does not surprise me that Gillings in his general work on Egyptian mathematics made no real effort to untangle the pefsu problems of the Moscow Papyrus (see below in the discussion of Problem 21 of the Moscow Papyrus). Other efforts to delineate and clarify the eight problems in which there are references to "1/2 1/4 malt-date beer" (and which are also not convincing to me) were later given by A.H. Gardiner, *Ancient Egyptian Onomastica*, Vol. 2 (Oxford, 1947), pp. 225-27, and by C.F. Nims, "The Bread and Beer Problems in the Moscow Mathematical Papyrus," *JEA*, Vol. 44 (1958), pp. 56-65. From the latter (p. 63) I have adopted one conclusion, namely, that the references to *bš3* ("grain") found in the beer problems were not to "spelt," as Struve wished, but rather to "malt," i.e., "sprouted grain."

[8] Compare the translation, transcription, and discussion of Problem 6 by Gunn and Peet, *op. cit.*, pp. 168-71 and Plate XXXV.

[9] See the discussion of this word in Gunn and Peet, *ibid.*, pp. 170-71. They feel that this is the square khet or aroura, as it is often translated, which appears in the Rhind Papyrus and elsewhere but in a different form than that found in the Moscow Papyrus. If indeed the setjat is meant here, then obviously the length and breadth of the rectangle in this problem are 4 khet and 3 khet.

[10] As Gunn and Peet note (*op. cit.*, p. 170, n. 1): "The word for 'square root' is written, here as elsewhere, with the sign , which represents either a 'corner' or more probably a 'right angle.' The underlying idea is perhaps that a right-angle with equal arms, say of 3 in length, ..., is the root of, in the sense of giving the *data* for, a square of [area] 9."

[11] See Chapter Four, the section on geometry. Of course, in modern terms, the solution is essentially the finding of the two unknowns (length and breadth of a rectangle) when two simultaneous equations are given: [1] 12 = length x breadth and [2] breadth = 3/4 of length. The value of the breadth in terms of the length which is given in equation [2] is substituted in equation [1], resulting in length^2 = 16 and length = 4. Then this value of length is substituted in equation [2], resulting in breadth = 3. Finally these numerical values of length = 4 and breadth = 3 are proved to be correct by multiplying them together to yield 12, the given area.

[12] Again see Chapter Four, section on geometry. Also note that Gunn and Peet, *op. cit.*, pp. 171-74, and Plate XXXV, give a translation, a hieroglyphic transcription, and discussion of Problem 7. Again we see some differences in the ways that Gunn and Peet and Struve treat the problem. See also Document IV.1, note 68, for a discussion of the various views of the Egyptian determinations of the areas of triangles.

[13] Gunn and Peet, *op. cit.*, p. 172, remark: "Lines 1 and 2. The determinative of *špdt*, 'triangle,' here, as in Problem No. 17 and the damaged Problem No. 4, has a shape quite different from that of the sign with which the same word is

written in the Rhind Papyrus. In the latter the sign is the symmetrical upright *point* (thorn?), with apex at top, with which all forms and derivatives of *špd* (primarily meaning 'to be sharp') are normally written in hieroglyphic and hieratic. In the Moscow Papyrus, on the other hand, it is a different sign, a scalene *triangle*, with vertical 'base' and the apex high up on the right. Thus the word *špdt*, 'the pointed,' in its special meaning of 'triangle' here receives a new determinative, a triangle." I again remind the reader of the discussion by these authors (*ibid.*, pp. 173-74) of their belief that the Egyptians had the proper formula for the area of a triangle (a correct belief, I think; see the section on geometry in Chapter Four and also note 14 below). The discussion makes some interesting philological points concerning *mryt* and *idb*, which they believe mean respectively "length and ratio of length to breadth" (to be interpreted in triangular areas as "height" and "base").

[14] This problem is here solved by the use of false position and the concept of proportion (see the section on geometry in Chapter Four). Of course, as in the preceding problem, the solution in modern terms is essentially one of finding two unknowns (the height and base of a triangle) with two simultaneous equations: [1] 20 = (1/2) height x base, and [2] height = 2 1/2 times the base. If there ever was any doubt that the ancient Egyptians knew the correct formula for a triangle, this problem should dispense with it. Notice that when the area is given, the first step in the solution in the finding of the height and the base is to double the given area of the triangle! From that point on it is clear that we are finding the sides of a rectangle that are equivalent to the *height* and *base* of the triangle! See my discussion of a scalene triangle with these measurements in the section on geometry in Chapter Four.

[15] This is a duplicate of Problem 5, except that the name of the weak beer is given in full here as "1/2 1/4 malt-date beer," while there it is abbreviated as "1/2 1/4 beer." See note 7 to Problem 5.

[16] See note 6 above.

[17] As in Problem 5, which concerns an exchange of bread for beer while this problem poses the use of Upper Egyptian grain to produce 100 loaves of bread and six 1-des-jugs of three beers of varying strength, i.e., with pefsu respectively of 2, 4, and 6, I have diverged from the attempt of Struve to clarify the problem; hence my translation differs from his. I found the comments of Peet in his review of Struve's text and translation (pp. 156-57) heipful but not definitive.

[18] As I have said in the body of Chapter Four (section "Volumes"), if Struve is correct in his view that the author is showing how to calculate the surface area of a hemisphere, this is a remarkable step in the development of geometry, a step ordinarily attributed to the Greeks (especially to Archimedes). It would

mean that like the Greeks, the Egyptians began to see the importance of calculating the curved surfaces of solids like the sphere in terms of a rectangular or rectiplanar surfaces. There are however some serious philological questions raised by Peet in his article "A Problem in Egyptian Geometry," *JEA*, Vol. 17 (1931), pp. 100-06. See especially pp. 101-03:

"Unfortunately there are against Struve's translation a number of very grave objections. The critical words are those which describe the *nbt* or 'basket':

nbt m tp-r r 4 1/2 m ʿḏ.

These words he translates 'einen Korb mit einer Mündung zu 4 1/2 in Erhaltung,' 'a basket with a mouth of 4 1/2 in preservation.' That *tp-r*...means 'mouth' seems very probable. *ʿḏ* Struve takes as the infinitive of the verb 'to be sound' or 'unharmed,' and thinks it indicates that the mouth, *tp-r*, is undamaged or complete, i.e., that it is, in technical language, not a small circle of a sphere but a great circle, and that consequently the *nbt* is a hemisphere and not a smaller segment of a sphere. To this there are fatal objections. The words *m ʿḏ* cannot possibly refer to *tp-r*, from which they are separated by *r* 4 1/2; in any case *m* with the infinitive cannot be attached adjectivally to a noun. Moreover, the expression *nbt m tp-r* 'a basket with a mouth...' is very doubtful Egyptian; the examples quoted by Struve from Pap. Anastasi I, 14, 3 and Pap. Harris I, 59, 2 are not parallel to this, for in both cases the dimension is followed by a genitive giving the figure, which is missing here. But the real rock on which Struve's rendering breaks up is the preposition *r* before the numeral 4 1/2. *r* is never used in the mathematical papyri to introduce a dimension when only one dimension is given; it is, however, used to introduce the second of two dimensions when two are given, and it then answers exactly to our 'by' in '6 feet by 3.' And this gives us the clue to the correct interpretation of the passage. The fignre 4 1/2, preceded as it is by *r*, must be the second of two dimensions. Where then is the first? It must be contained in the 9 which so unexpectedly turns up without explanation in line 5, where its sudden appearance is so disconcerting to Struve. But why was this not mentioned in its proper place in the setting out of the problem? The answer is that in the archetype it was, but that our scribe has omitted it. I am convinced that no one who is conversant with the phraseology of the mathematical papyri and with the Middle Egyptian uses of the prepositions will question the necessity of inserting the word ⌓ followed by a numeral between *nbt* and *m* in line 2, thus restoring the reading

nbt <nt x> m tp-r r 4 1/2 m ʿḏ

'a basket (?) of x in mouth and 4 1/2 in *ʿd*' where *ʿd*, whatever it may mean, is the name of the second dimension given, just as *tp-r* is of the first. The working now becomes intelligible. Two dimensions are given; the first is operated on in lines 5ff., and the second, the 4 1/2, is only brought in near the end as a multiplier.....

"Once the grammatical necessity for restoring these two words is perceived Struve's interpretation of the problem as the determination of the curved area of a hemisphere of diameter 4 1/2 falls under the gravest suspicion, for a hemisphere is fully determined by a single dimension, its radius or diameter, while here we have two, a *tp-r* and a *ʿd*.

"To this it may be replied that Struve has produced strong etymological evidence to show that the *nbt* is in effect a hemisphere. This evidence we must now examine. The word ⌂ |, the reading of which as *nbt* seems certain, is doubtless in origin the well-known word for a 'basket,' as Struve has pointed out, and our first instinct is naturally to see in it the technical term for a hemisphere, or at any rate a segment of a sphere, which its shape suggests. Struve, who translates it as hemisphere, finds confirmation of this in line 6, where he thinks that the *nbt* was stated to be half an *inr*, 'egg,' which he holds to be the technical term for a sphere."

Peet goes on to refute that view, and concludes that *inr* cannot be used alone for an "egg" or an "egg-shell" and hence "I do not accept the reading *inr*; and with this reading falls the etymological argument for Struve's interpretation of the problem." Examination of the hieratic text makes me hesitant to accept the last argument concerning *inr*. There is no doubt in my mind that the word which Struve read as *inr* has at its end the egg glyph, and one should not dismiss the example of its use of *inr* as an "egg-shell" in a hymn cited by Struve simply because it was a figurative use, since figurative uses often become conventional usage. The reading of the word itself is too fragmented to sustain the doubts raised by Peet. Though Gillings in his treatment of Problem 10 (*op. cit.*, pp. 198-201) does not answer Peet's philological dismissal of Struve's interpretation, he was inclined to accept Struve's interpretation of the problem as the finding of the curved surface of a hemisphere. He does so because a step-by-step consideration of the arithmetical procedure when generalized seems to produce the modern form of the formula for the curved surface of a hemisphere. Numbering the lines consecutively rather than by column, on page 199 he correlates the steps with the modern procedue:

"(Line 5). Double 4 1/2. Double the diameter $=2d$.

(Lines 6,7). Find 8/9 of this. $8/9 \times 2d = 2 \times 8/9 \times d$.

(Lines 8,9,10,11). Find 8/9 of this. $2 \times 8/9 \times 8/9 \times d$.

(Line 13). Multiply by d. $2 \times 64/81 \times d^2 = 2 \times 64/81 \times (2r)^2$, or
$A = 2 \times 256/81\, r^2$
$A = 2\pi r^2$, where $\pi = 256/81$.

This is indeed the modern formula for the curved surface of a hemisphere. If this interpretation of MMP 10 is the correct one, then the scribe who derived the formula anticipated Archimedes by 1500 years!"

Even if this is the correct interpretation, there is in this numerically presented formula (as in all the arithmetically expressed geometrical formulas given by Egyptians) no formal derivation and proof of the formula implied by the arithmetical steps. As Gillings suggests (p. 200-01), the procedure could have been derived empirically by the basket weaver who found that "when one is weaving baskets which are roughly hemispherical one requires a quantity of material for the circular plane lid that is almost half that required for the basket itself. Since the calculation of the area of a circle was a commonplace operation to the scribes (Problem 50 of the RMP [my Document IV.1]), over a period of years it could have come to be equally commonplace that the curved area of the hemispherical basket was double that of the circular lid." But though I tend to agree with Struve and Gillings, I must remind the reader that the proponents of this interpretation of Problem 10 must still meet some of the philological difficulties raised by Peet, as I have partially done earlier in this note.

19 This is one of two alternate interpretations posited by Peet. I have not taken seriously his reconstruction of the problem as determining the area of a semicircle, since a semicircle as a flat surface can hardly be considered as a "basket" and the problem, whatever its exact intention, certainly seems to concern the area of some kind of basket. I am not persuaded by the remaining interpretation given next. It seems to me there are even more gratuitous assumptions in it than there are in Struve's, and Peet himself points out some of the difficulties. Perhaps the most cautious conclusion is that, in view of the state of the text, we cannot be sure of the correct interpretation.

20 Peet in his review of Struve's edition (p. 158) makes a number of crucial emendations of Struve's transcriptions in this problem: "P. 101, no. 11. S. has gone hopelessly wrong here through an incorrect transcription. The numeral 100 occurs three times in the problem and he has on each occasion failed to recognize it; in xxi, 3 he has read it as , in xxii, 2 as , and in xxii, 3 as . Further alterations to be made in S.'s readings are as follows: Read for *passim*. In xxi, 4-6 read and for and respectively. In xxii, 3 for read . The real difficulty lies in the word with which xxi, 4 and 5 begin, and which occurs again in xxii, 4. For the first

sign, is palaeographically just possible, though a comparison with the other instances in this papyrus shows that this would be an extreme form, and in view of the very different form given to in writing *pḫdt*, where, as here, it stands alone, the reading , *šȝ*, is preferable. This is confirmed by the writing with phonetic complement in xxii, 4. In the ligature which follows the top sign might be *r* or *t* (*d* is mostly better made) and the lower *r*, *t*, or *n*. The determinative looks like ." Following these remarks, Peet gives the translation which I have followed closely in meaning, though maintaining some of the phraseology I have adopted in translating the previous problems.

[21] I am following the tenor of the remarks made by Peet in his review (*op. cit.*, p. 157). Needless to say, the pefsu is, as I have said before, a ratio of quantities and thus a simple number. Note that the remainder of the calculation to find the pefsu is given in numbers alone without specification of des-jugs and/or heqat.

[22] As Struve shows in his edition (p. 92), the calculation given by the author is similar to solving the linear equation x = 18: (13: 13/6) in two steps: (13: 13/6) = 6 and 18:6 = 3 = x.

[23] This is a duplicate of Problem 9, except that it is considerably truncated. Notice also the scribal error in Col. XXVI, line 3, where "12" is given instead of "6."

[24] See Chapter Four under the rubric "Volumes" for an extended discussion of this correct formulation of the procedure for finding the volume of a truncated square pyramid, which is usually judged to be the most important volumetric discovery made by the ancient Egyptian mathematicians.

[25] For some philological comments by Peet, see his review, p. 156.

[26] Peet in his review (*op. cit.*, p. 157) criticizes a number of the readings and reconstructious of Struve. But he seems at a loss to offer any satisfactory translation of the whole problem. Hence I have stayed with Struve's text.

[27] I have kept a close eye on Gunn and Peet, *op. cit.*, pp. 174-75 and plates XXXV and XXXVI. Also see the corrected readings in Peet's review, *op. cit.*, p. 160. Note that the drawing in the papyrus is of a triangle that is almost a right-triangle and probably was meant to be that, as Struve redraws it in Fig. IV.6m, Col. XXXIV. I have already discussed in Chapter Four, the sections on areas, and in Document IV.1, note 68, the question of whether the general formula for the triangle was known by the Egyptians and I have expressed my opinion that it was. The area is specified as "two thousands of land," which is

20 setjat. I have simply used setjat for square khets throughout. Peet uses "arurae." It is also often written "arouras."

[28] For Peet's highly critical remarks concerning Struve's reading of the text of this problem and his translation of it, see his review of Struve's text *(op. cit.*, pp. 159-60). In the last sentence of these remarks, Peet figuratively throws up his hands and says: "All this seems hapeless."

[29] For the measured lengths of cubits and palms, see my Volume One, p. 109 and the first sections of Chapter Four above. It will be obvious to the reader that the problem, as interpreted here, is simply a standard rectangular area problem after the length, given as a mixed number of cubits and palms (or handbreadths, following Peet's rendering), is converted entirely into palms, while the breadth remains in palms. The resulting area, then, is in squared palms.

[30] See the brief but useful corrections in Peet's review (p. 159). It will be evident to the reader that the arithmetic solution to Problem 19 is equivalent to the solution of a simple linear equation $(1\ 1/2)x + 4 = 10$.

[31] Note that in line 5 the calculation is made in heqat with Horus fractious (which, as in Document IV.1, I have given in Italic type) plus a further fraction expressed in ro.

[32] While Peet in his review (pp. 157-58) believes that Struve's "interpretation of the nature of the problem is certainly right," he offers some corrections, which I have adopted in my translation. I have quoted Peet's later rendering of this problem in Chapter IV under the rubric "Pefsu Problems." Notice that the final answer, namely 1/12, is in fact the harmonic mean between 1/8 and 1/16. Gillings, *op. cit.*, p. 132, has an interesting comment on this fact: "In line 7 (=Col. XXXIX, line 2 of Struve's text) the scribe divided 5 by 60, where we would have expected 60 divided by 5 giving the pesu of the sacrificial bread as 12. If one hekat of grain produced 12 loaves of bread then each of these loaves would have a pesu of 12. But the scribe has expressed this differently by saying that each loaf contained one twelfth of a hekat of grain, which is correct. This method of expressing pesu appears to be consistent with line 2 [of Col. XXXVIII] where the fractions 1/8 and 1/16 are written for what we would call pesus 8 and 16. If then following the scribe's thoughts we think of fractions only, we come quite naturally to the observation that the answer 1/12 is the harmonic mean of the two fractious 1/8 and 1/16, being equal to twice their product divided by their sum." (Note that Gillings' system of writing unit fractions by inserting macrons over the denominators has been changed in my quotation to my conventional system of writing the fractions.)

[33] Again see that the one-half heqat is written in the Horus-eye fractional system, i.e., with its special sign for 1/2. Hence I have written it in Italics, as al-

ways. For stronger and weaker beers, see my version of Problem 5 above with note 7.

[34] The reconstruction of this problem by S. Couchoud, *Mathématiques égyptiennes* (Paris, 1993), pp. 171-74, also depends on Peet's suggestions for the most part. She makes a further clever correction in the translation in line 5 of *rmnʿwy* (*i.e.*, *r mn ʿwy?*)...*10* as "up to these 10 pairs." She quotes *Wb*, Vol. 1, p. 158/12. I have adopted the translation "equivalencies" (see R.O. Faulkner, *A Concise Dictionary of Middle Egyptian* [Oxford, 1962], p. 149) because of the somewhat awkward word order and the rather meaningless expression "up to" for *r mn* involved in accepting Couchoud's suggestion. This is avoided by translating the term as "[day-] equivalencies" or "equals" or "matchings," meaning, of course, the two different day-amounts necessary for, on the one hand, the cutting and, on the other, the decorating of the ten pairs of sandals.

[35] It is evident from the details of this problem that 10 of the 15 heqat were used to make the 200 loaves of bread of pefsu 20, and that 5 heqat were used to produce the 10 des-jugs of beer of pefsu 2. This, like all of the pefsu problems, involves the use of the basic cooking ratio of the number of loaves of bread or jugs of beer to the quantity of flour or grain.

[36] For corrections in the first phrase of line two, see Peet's review (p. 159), where he says "the original reading was doubtless *in m pꜣ ʿḥʿ*," and then gives the translation I have included here. The problem expressed in algebraic form is $2x + x = 9$, and x is, of course, 3, as the author determines.

DOCUMENT IV.3

The Kahun Mathematical Fragments: Introduction

Among the finds that resulted from Flinders Petrie's excavations at the workers' town (designated as Kahun by Petrie) near the pyramid of Sesostris II at Illahun in 1889-90 were a number of hieratic texts and fragments. These were published in 1899 in two volumes (Vol. 1, Text and Vol. 2, Plates) by F. Ll. Griffith, whose articles on the Rhind Papyrus I have already mentioned. The title was *The Petrie Papyri: Hieratic Papyri from Kahun and Gurob*, 2 vols. (London, 1898).

Among the fragments were six devoted to mathematical problems. So far as he was able, Griffith translated and discussed them (for the hieratic texts and hieroglyphic transcriptions, see Fig. IV.12, which reproduces Plate VIII of Griffith's work) and my English translations have been made from his text and, in part, are based on his suggested translations. The fragments seem to date from the second half of the twelfth dynasty, and so are roughly contemporary with the first two documents that I have presented.

In the preface to the text-volume Griffith gives us some of the details of when and how he developed his work (pp. v-vi):

> The restorations of the papyri were mostly completed in 1890. It was at first intended to publish hand copies...[but it was later] agreed that the facsimile plates should be made by the excellent photographic process of the Autotype Company. These facsimiles were all executed in 1893.... At length, in March, 1897, Part I., containing Plates I. to VIII. [and hence the mathematical fragments], and the text pertaining to them, was issued to subscribers.... The subscription copies were accompanied by a request to scholars for corrections and observations, and these the editor has now the pleasure of acknowledging in the section of *Additional Notes* [the notes for Plate VIII. by Professor Maspero and

> others are included on p. 101]...and in the *Zeitschrift für Aegyptische Sprache*, 1898, Borchardt has contributed to the solution of a mathematical problem.

I have included this group of the so-called Kahun-Papyri fragments, despite their apparent lack of anything substantially different from the problems and techniques of the first two documents, because they do give one more small indication of the popularity of some of the tables and problems we have already presented.

The first item from these fragments contains a fifth of the Table of Two, which appeared in more complete form in Document IV.1 (the Rhind Papyrus). The initial item is from the fragment Kahun, IV.2, lines 1-10, and gives the divisions of 2 by the first ten odd numbers, i.e., those from 3 through 21. The answers are expressed in the same unit fractions as those given in the complete table of the Rhind Papyrus, though they are much more simply presented as 10 lines of numbers with no check marks and no rubrication.

The second fragment is from Kahun IV. 3. (vertical lines or columns 11-12) and contains numbers in arithmetical progression. It involves the following problem: "When the sum of 10 terms in arithmetical progression is given and the common difference of those terms is also given, what is the series?" This resembles Problem 40 of the Rhind Papyrus.

The next item is also from Kahun, IV. 3. (columns 13-14). It concerns the volume of a cylindrical granary, and thus is a type of problem found in Document IV.1 (see Problem 41-43). It is important because it gives the procedure of finding the volume of the cylinder in khar directly and thus allows us to correct the similar Problem 43 in Document IV.1.

The fourth item is from Kahun, XLV. 1. (lines 15-22). It contains a group of large numbers, the context being unspecified. Griffith's comment concerning these numbers (*op. cit.*, p. 16) still seems just:

> These fragments (*verso* blank) are useful as showing the hieratic forms of the highest numerals. I do not see what the very large quantities mentioned are likely to refer to: they diminish rapidly in succeeding lines, but apparently not in any fixed proportion. Yet it seems probable that they formed part of a considerable mathematical calculation, and not of mere accounts.

The next problem I have included for Document IV.3 is from Kahun, LV. 3. (lines 23-28). It is simply an aha-problem, i.e., one whose specified conditions and the solution resemble those of a linear equation whose unknown quantity is to be found. As we have seen, this is also a type of problem found in the other documents (e.g., see Document IV.1, Problems 24-29).

The remaining two extracts that comprise the remainder of Document IV.3 come from Kahun LV. 4. (lines 30-62). They are embraced by the general title: "Example of calculating the problems (?) of account-keeping." The first problem is entitled "A Calculation" by Griffith; and apparently involves rectangular areas with sides expressed in cubits, though a mention of henu (a pint measure) is puzzling. The second is "Account of the produce [of fowls?]" and involves ducks, geese, and cranes.

DOCUMENT IV.3

The Kahun Mathematical Fragments

[A Table of Two; Kahun, IV.2; see Fig. IV.12, lines 1-10]

[I have first given the unadorned numbers of each line as they appear in the papyrus, but I have added commas to separate the numbers. I have followed each string of numbers with a bracketed modern interpretation. Note that the first number of each line is understood to be 2. It is given in the first line and each line beneath it is indented, presumably to indicate the same number 2 for all the remaining lines. Hence, I have added a bracketed 2 in each line.]

[Lin. 1] 2, 3, 2/3, 2 [i.e., 2:3 = 2/3 since 2/3 of 3 = 2].

[Lin. 2] [2,] 5, 1/3, 1 2/3, 1/15, 1/3 [i.e., 2:5 = 1/3 + 1/15, since 1/3 of 5 = 1 2/3, 1/15 of 5 = 1/3, and 1 2/3 + 1/3 = 2].

[Lin. 3] [2,] 7, 1/4, 1 1/2 1/4, 1/28, 1/4 [i.e., 2:7 = 1/4 + 1/28, since 1/4 of 7 = 1 1/2 1/4, 1/28 of 7 =1/4, and 1 1/2 1/4 + 1/4 = 2].

[Lin. 4] [2,] 9, 1/6, 1 1/2, 1/18, 1/2 [i.e., 2:9 = 1/6 + 1/18, since 1/6 of 9 = 1 1/2, 1/18 of 9 = 1/2, and 1 1/2 + 1/2 =2].

[Lin. 5] [2,] 11, 1/6, 1 2/3, 1/6, 1/66, 1/6 [i.e., 2:11 = 1/6 + 1/66, since 1/6 of 11 = 1 2/3 1/6, 1/66 of 11 = 1/6, and 1 2/3 1/6 + 1/6 = 2].

[Lin. 6] [2,] 13, 1/8, 1 1/2, 1/8, 1/52, 1/4, 1/104, 1/8 [i.e., 2:13 = 1/8 + 1/52 + 1/104, since 1/8 of 13 = 1 1/2 1/8, 1/52 of 13 = 1/4, 1/104 of 13 = 1/8, and 1 1/2 1/8 + 1/4 + 1/8 =2].

[Lin. 7] [2,] 15, 1/10, 1 1/2, 1/30, 1/2 [i.e., 2:15 = 1/10 + 1/30, since 1/10 of 15 = 1 1/2, 1/30 of 15 = 1/2, and 1 1/2 + 1/2 = 2].

[Lin. 8] [2,] 17, 1/12, 1 1/3 1/12, 1/51, 1/3, 1/68, 1/4 [i.e., 2:17 = 1/12 + 1/51 + 1/68, since 1/12 of 17 = 1 1/3 1/12, 1/51 of 17 = 1/3, 1/68 of 17= 1/4, and 1 1/3 1/12 + 1/3 + 1/4 = 2].

[Lin. 9] [2,] 19, 1/12, 1 1/2 1/12, 1/76, 1/4, 1/114, <1/6> [i.e., 2:19 = 1/12 + 1/76 + 1/114, since 1/12 of 19 = 1 1/2 1/12, 1/76 of 19 = 1/4, 1/114 of 19 = 1/6, and 1 1/2 1/12 + 1/4 + 1/6 = 2].
[Lin. 10] [2,] 21, 1/14, 1 1/2, 1/42, 1/2 [i.e., 2:21 = 1/14 + 1/42, since 1/14 of 21 = 1 1/12, 1/42 of 21 = 1/2, and 1 1/2 + 1/2 =2].

[Kahun Papyrus IV. 3.; see Fig. IV.12, Cols. 11-12]

[It seems evident that the problem being solved in the two columns of numbers is: "Given the sum of 10 terms in arithmetic progression as 100 and the common difference of terms as 1/2 1/3, what is the series?" [1]]

[Col. 11]		[Col. 12]
\ 1	1/3 1/12	100 [items to be divided among] 10 [men] [in arithmetically decreasing amounts]
2	2/3 1/6	13 2/3 1/12
4	1 2/3	12 2/3 1/6 1/12
\ 8	3 1/3	12 1/12
Total	3 2/3 1/12	11 1/6 1/12
		10 1/3 1/12
		9 1/3 1/6 1/12
		8 2/3 1/12
		7 2/3 1/6 1/12
		7 1/12
		6 1/6 1/12

[Kahun Papyrus IV. 3.; see Fig. IV.12, Cols. 13-14]

[What is the volume in khar (1 khar = 2/3 a cubic cubit) of a cylindrical granary whose diameter is 12 cubits and whose height is 8 cubits? The procedure is to add 1/3 of the diameter to the diameter, multiply the total by itself; then multiply that result by 2/3 of the height, i.e., 5 1/3, to produce 1365 1/3 khar. I put the operations of Col. 14 first].

[Col. 14][2]

[\ 1	12]
2/3	8

\ 1/3   4
Total   16.

\ 1   16
\ 10   160
\ 5   80
Total 256.

[Col. 13]
\ 1   256
2   512
\ 4   1024
\ (1/30)* 85 1/3   [*2/3 in Papyrus]
Total   1365 1/3 [khar].

[Kahun Papyrus XLV. 1.; see Fig. IV.12, lines 15-22]
[The following lines present 7 large numbers and a fraction. Their context is not known.[3]]
[Lin. 15] 925,157 1/3
[Lin. 16] 708,453 1/3
[Lin. 17] 709,533 1/3
[Lin. 18] 500,098 2/3 1/8 1/16
[Lin. 19] 470,042 2/3
[Lin. 20] 440,003 1/6
[Lin. 21] 209,200
[Lin. 22]   1/12.

[Kahun Papyrus LV. 3.; see Fig. IV.12, lines 23-28][4]
[Lin. 23] 1/2 [of a quantity] minus (?) 1/4 of it yields 5.
[Lin. 24] What number says it (i.e., satisfies the statement)? Produce [a remainder of] 1
[Lin. 25] after 1/4 is subtracted from 1/2. The result of it is 1/4. Calculate with 1/4
[Lin. 26] to find 1. The result is 4 times.
[Lin. 27] Take 5, 4 times. The result is 20.

[Lin. 28] [Hence] it is 20 that says it (i.e., satisfies the initial statement).[5]

[Kahun Papyrus LV. 4.; see Fig. IV.12, lines 30-42]

[This fragment is on the first of three pages. The problem is incomplete in both its beginning and its end. It uses the word for square root (*ḳnbt*, i.e., "corner" or "right angle") in line 40, also found in the Moscow Mathematical Papyrus.[6] I have followed the text of Griffith for the most part, but have departed from his translation in line 42.[7]]

[Lin. 30] **Example of the calculating of problems (?) of account-keeping.**

[Lin. 31] ....

[Lin. 32] ...

[Lin. 33] ...of the henu (pint measure) ?....

[Lin. 34] Take 40, 3 times.

[Lin. 35] The result is 120. Take

[Lin. 36] 1/10 of 1[20]. The result is 12.

[Lin. 37] Calculate with 1/2 1/4 to find 1.

[Lin. 38] The result thereof is 1 1/3. Take

[Lin. 39] 12, 1 1/3 times. The result thereof is 16.

[Lin. 40] Take the square root [of it], which is 4. Take

[Lin. 41] 1/2 1/4 of 4. The result is 3.

[Lin. 42] The result is 10 rectangles (*ḥ3yt*) of 4 cubits by 3.

[Kahun LV. 4.; see Fig. IV.12, lines 43-54]

[This is the second page of the fragments of this papyrus; the third page (lines 55-62) is so incomplete that I have not added it, though it is given in its incomplete state by Griffith and can be found on Fig. IV.12. The account given here is not very clear, as Griffith notes, but he concludes: "In l. 54 it is perhaps found that 11 birds, each of an average value of 5 *set*-ducks, would balance the account: p. 3 [too fragmentary to include here] may then have showed how these were best to be distributed amongst the different kinds."[8] The reader will note that Document IV.1, Problem 83, concerns the feed necessary for several similar birds. The final column in the translation under 'Total value' is not present in the papyrus, but is suggested by the calculations of lines 50-54. I have followed Griffith in adding the headings to the columns.]

[Lin. 43] **Account of the produce [of fowls (?)].**
[Lin. 44] List (*rḫt*) of the produce of 100 [Set-] ducks.
[Lin. 45] Paid to him from amongst this list:

		[Value of each one in Set-duck]	[Number of each]	[Total value]
[Lin. 46]	Re-goose	8	[3]	[24]
[Lin. 47]	Terp-goose	4	[3]	[12]
[Lin. 48]	Djendjen crane	2	[3]	[6]
[Lin. 49]	Set-duck	1	[3]	[3]
		[Totals:]	[12]	[45]

[Lin. 50] 1 is subtracted from the number of fowl;
[Lin. 51] the remainder is 11. Calculate the excess of 100
[Lin. 52] over 45. The result thereof is 55. Do
[Lin. 53] the multiplication of 11 to find 55.
[Lin. 54] The result thereof is 5 times.

Notes to Document IV.3

[1] The important correction to Griffith's reading of the first item in column 12 is that made by S. Couchoud, *Mathématiques égyptiennes* (Paris, 1993), p. 164: "100 10" instead of Griffith's 110. Everything falls into place if this reading is accepted. The suggestion by Gillings, *Mathematics in the Time of the Pharaohs*, pp. 178-80, which accepts 110 as the sum, and accordingly a series of 12 terms, but without the last two specified, is less probable, and I feel sure is to be rejected. So also to be rejected is his earlier article: "Mathematical Fragment from the Kahun Papyrus," *Australian Journal of Science*, Vol. 29, No. 5 (1966), pp. 126-30. But see his useful discussion of the steps given in RMP Problem No. 64 in his short "Sum of *n* Terms of an Arithmetical progression in Ancient Egypt," *Australian Journal of Science*, Vol. 31, No. 1 (Sydney, 1968),

pp. 47-48. Gillings' comments as to what the Egyptians could have done with their knowledge of arithmetical progressions are somewhat fanciful.

Going back to the text as given, it should be noted that the highest term h in the series is determined precisely by the steps indicated by the formula: $h = (S/n) + (n-1)(d/2)$, where S is the sum of the terms (100), n is the number of terms (10), and d is the common difference between terms (2/3 + 1/6). Hence Col. 11, starting with $d/2$, *i.e.*, 1/3 + 1/12, shows that when it is multiplied by $n-1$, i.e., by 9, the result is 3 2/3 1/12. Then, moving on to Col. 12, S/n, 100 /10 becomes 10 and when added to the result of Col. 11, yields the highest term, namely 13 + 2/3 + 1/12. The succeeding terms were then determined by subtracting the common difference. The reader should compare this problem with Problem 64 of the Rhind Papyrus.

[2] For a discussion of this problem, see Chapter 4, section "Volumes." Note that in Col. 14 in Fig. IV.12 a small circle is drawn preceding the numbers. It is marked with 12 (the diameter) at its top and 8 (the cylinder's height) on its left. In the last calculation of Col. 13, the multiplier should be "1/3," as I have given it. The papyrus erroneously has "2/3."

[3] As far as I can judge, neither an arithmetical nor a geometrical series is eviudent in these numbers as written. One trivial point can be noted. The number in line 15 is the only one of the long numbers that is indented.

[4] This is a problem involving an unknown of the type we have seen often in the first two documents. It is like solving the equation $(1/2)x - (1/4)x = 5$ by first assuming that x is 1 and then converting the false remainder (1/4) to the true remainder (5), i.e., finding the multiplier of 1/4 that produces 5. The answer is of course 20.

[5] A further fragment is given as line 29: "Example of proof (*tp n šyty*)," an expression also found often in the Rhind Papyrus (e.g., Document IV.1, Problems 32-35, 37-38).

[6] See Document IV.2, notes 10 and 11.

[7] For the translation of *ḥ3yt* as "rectangle," see Schack-Schackenburg, "Der Berliner Papyrus 6619," *ZÄS*, Vol. 38 (1900), p. 137. This translation is found in *Wb*, Vol. 3 (1971), p. 15, item 19, but with a "?" following it. Cf. S. Couchoud, *op. cit.*, pp. 138-39. Line 33 remains obscure. It may be that the problem involved a volume, but the remaining lines point to a problem concerning the areas of rectangles. Note that Couchoud questions Griffith's reading "henw" in line 33. If it is not a volumetric measure that is being used, then we need not find a more complex volumetric interpretation, as does Schack-Schackenburg (*op. cit. supra*, pp. 129-30) and Gillings, *op. cit.*, pp. 162-65 (who suggests it is a problem of simultaneous equatious of the form $xy = A$ and $x = ky$).

[8] Griffith, *Hieratic Papyri from Kahun and Gurob*, p. 18.

DOCUMENT IV.4

The Berlin Papyrus 6619: Introduction

Judging from their nature, the mathematical fragments treated here as Document IV.4 were apparently composed about the same time as the Kahun and Moscow Mathematical Papyri, namely, sometime from the second half of the 12th dynasty through the 13th dynasty. The Berlin fragments were first presented by Hans Schack-Schackenburg in two articles: "Der Berliner Papyrus 6619," *ZÄS*, Vol. 38 (1900), pp. 135-40 and Tafel IV.1, and "Das kleinere Fragment des Berliner Papyrus 6619," *ibid.*, Vol. 40 (1902), pp. 65-66. Concerning the first problem (see Fig. IV.13, 1), he was able to reconstruct its basic purpose and content, as Archibald neatly indicated in his extensive Bibliography of Egyptian Mathematics appearing in Volume 1 of Chace's edition of the Rhind Papyrus under the year 1900:

> The problem here referred to may be stated thus: Distribute 100 square ells [i.e., square cubits] between two squares whose sides are in the ratio 1 to 3/4; whence the equations $x^2 + y^2 = 100$, $x : y = 1 : 3/4$, corresponding to those given in Griffith (1897). The equations are solved by the method of false position and the solution of two term quadratic equations. On the back of this fragment of the papyrus [see Fig. IV.13, 2] is another problem somewhat similar to no. 69 of the Rhind papyrus. A translation of both sides of this fragment is given by A. Erman and F. Krebs, *Aus den Papyrus der königlichen Museen*, Berlin, 1899, pp. 81-82, "Aus einem Rechenbuch;" but the authors acknowledged their inability to give an explanation.

Under the year 1902, Archibald also briefly characterizes the smaller fragment of the same papyrus (see Fig. IV.13, 3):

We have here a problem similar to that in the larger fragment, and leading to the equations $x^2 + y^2 = 400$, $x : y = 2 : 1\ 1/2$.

Gillings in his oft-cited *Mathematics in the Time of the Pharaohs*, pp. 161-62, also points out the two sets of simultaneous equations which embrace the data and the arithmetical steps of the two fragments. In addition, he gives a free English translation, without bracketed additions, of the German rendering of the longer problem proposed by Schack-Schakenburg (see note 1 to the document below) and a summary of the shorter fragment based on the German author's second article.

After the original transcription and translation of these fragments by Schack-Schackenburg, the most important reexamination of the problems is that of Sylvia Couchoud in her perceptive work: *Mathématiques égyptiennes* (Paris, 1993), pp. 131-34 and 142-43. In regard to the longer fragment, she presents "some new points of view on the transcription and translation." I have taken them into account in my translation.

DOCUMENT IV.4

The Berlin Papyrus 6619

[The Longer Fragment; see Fig. IV.13, 1; Fig. IV.14a; and Fig. IV.15a][1]

[Lin. 1] **Another [example of dividing a given rectangular (i.e., square) area of 100 (square cubits) into two smaller squares.]**[2] If someone says to you: ["100 square cubits is divided] into [2] unknown [square surface-] quantities (*ꜥḥꜥw*) [and 1: 1/2 1/4 is the ratio of the side of]

[Lin. 2] the first quantity to that of the other quantity, please make known to me the unknown [surface-]quantities."

[Lin. 3] The calculation of [one of the] rectangles is with 1 always and the calculation of the other is with 1/2 1/4 of 1; [the result of 1/2 1/4 of 1 is 1/2 1/4. Take]

[Lin. 4] the 1/2 1/4 of the [side length] of the one surface quantity (i.e., square) for the [side of the] other. The result is 1/2 1/4. Multiply it by [1/2 1/4. The result is 1/2 1/16 for the area of the smaller square surface].

[Lin. 5] [Hence] if the quantity of the [side of the larger] square is 1, and that of the other is 1/2 1/4, [and] you will take the sum [of their squares,]

[Lin. 6] the result is 1 1/2 1/4 *(delete)* 1/16 (i.e., 25/16). You will take its square root. The result is 1 1/4. You will then take [the square root of 100].

[Lin. 7] The result is [10]. Reckon with this 1 1/4 to find 10 (i.e., find the multiplier of 1 1/4 that yields 10). The result is the quantity 8 [for the side of the larger square].

[Lin. 8] [You will take 1/2 1/4 of] this 8. The result is [the quantity 6 for the side of the smaller square].

[The Shorter Fragment; see Fig. IV.13,3, Fig.14b, and Fig. 15b]

[Presumably the missing problem-title was presented in a form somewhat like the following: **Example of dividing a square rectangular surface area of 400 square cubits into two square areas whose sides were related as 2 : 1 1/2.** Then on the basis of the procedure in the longer fragment, the method of false position would dictate the assumption of the side of the larger square as 2 and that of the side of the smaller square as 1 1/2, and thus the squares as 4 and 2 1/4 and their sum as 6 1/4. Then follows the extant text of the fragment:]

[Lin. 1] ...You should extract the square root of 6 1/4...[i.e., 2 1/2]
[Lin. 2] ...[Take] this 2 1/2, which remains....[You take]

[Lin. 3] [...the square root of 400, i.e., 20]. Reckon [with 2 1/2 to obtain 20]...[The result is 8] times. [Multiply 8]
[Lin. 4] ...[by 2 and 1 1/2.] You should [now] say to him, the square root[s]
[Lin. 5] ...[of the component square]s according to this calculation (*irt*) [are 16]
[Lin. 6] ...[and] 12. You say it is found ...[i.e., correctly?][3]

Notes to Document IV.4

[1] I follow the line-numeration of Schack-Schackenburg. Notice that in Fig. IV.13,1 Schack-Schakenburg provides the hieratic text of the longer fragment, and that Fig. IV.14a gives S.-S.'s hieroglyphic transcription of that text while Fig. IV.15a presents S. Couchoud's transcription and French translation. I note further, as I have already said in the Introduction to the document, that Gillings (*op. cit.*, p. 161) gave a free translation of S.-S.'s German rendering. It follows:
"If it is said to thee...the area of a square of 100 [square cubits] is equal to that of two smaller squares.
The side of one is 1/2 1/4 the side of the other. Let me know the sides of the two unknown squares.
Always take a square of side 1. Then the side of the other is 1/2 1/4.
Multiply this with 1/2 1/4. It gives 1/2 1/16, the area of the small square.
Then together, these two squares have an area of 1 1/2 1/16.
Take the square root of 1 1/2 1/16. It is 1 1/4.
Take the square root of this 100 [square] cubits. It is 10.
Divide this 10 by this 1 1/4. It gives 8, the side of one square.

The remainder is very much damaged, but what does remain leads Schack-Schackenburg to restore [it] as[follows:]

Take 1/2 1/4 of these 8. It gives 6, the side of the other square."

Gillings notes that the procedure for taking the square roots of 1 1/2 1/16 and 100 is not given and he suggests: "If these were not done mentally, they were no doubt read off or checked from a table of squares (see Chapter 21). Neither is the working shown for 1/2 1/4 of 8 = 6, and we may justly conclude that the scribe was more concerned to show how his method of false position or false assumption was applied to the solution of equations than to teach the arithmetic of multiplication of fractions and the squaring and square roots of simple fractions." I note that no such Egyptian table of squares has been found. In my quotations from Gillings I have converted his form of expressing unitary fractions by the reciprocal notation with macrons above the numbers. It will be noticed that I have kept much closer to what is actually said in the text than the free version of Gillings. I do not claim that the expressions in brackets in my translation are anything more than informative additious to suggest what the author may have intended. Of course, that which remains unbracketed is what can be read in the surviving fragment. I recommend particularly that the reader keep a close eye on Couchoud's transcription and discussion of this fragment.

[2] This bracketed addition in bold type is very speculative; but, in view of the succeeding text that can be read, something like it was probably given.

[3] This is correct since $16^2 + 12^2 = 400$.

DOCUMENT IV.5

The Mathematical Leather Roll in the British Museum: Introduction[1]

The Mathematical Leather Roll at the British Museum was acquired by the Trustees of the Museum in 1864, having been a part of the collection of Egyptian antiquities assembled by A.H. Rhind in Thebes.[2] Its purchase followed Rhind's death. It will be recalled that the Rhind Mathematical Papyrus (Document IV.1 above) was also a part of that collection. But unlike that more famous work it was, as its commonly used name indicates, written on leather rather than papyrus. Consequently it was far more difficult to unroll, that not being accomplished until 60 years after the roll's purchase.[3] The early history of the roll after its acquisition by the Museum is described by Glanville.[4]

> The presence of the leather roll in the British Museum was common knowledge at the time of, or at any rate immediately after the publication of the first complete study of the [Rhind] papyrus, for Eisenlohr states the fact when describing the papyrus, and adds that the leather was too brittle to unroll. Some years later Professor Griffith saw the roll and recognized a fine hand in the beginnings of numerical signs which could be seen just inside the edge. So that although there was still no means of unloosening the coil, there were yet no grounds for the curious scepticism as to its actual existence on the part of one of the most learned of living Egyptologists [i.e., Gunn]. The question of unrolling was again brought up last year [1926?] by Professor Griffith, who had heard, in Berlin, of a new treatment for softening ancient leather. In the interval between his first sight of the roll and his chancing on this German process, that whole department of archaeology which consists in the "restora-

> tion and preservation of antiquities" had been organized, and the brilliant successes of Lucas in the Valley of the Kings were being matched at home by research in the laboratory attached to the British Museum, and by the regular Reports of its Director, Dr. Alexander Scott, F.R.S., on the museum objects submitted for treatment. It was therefore possible to reconsider the unrolling of the leather roll....Even with the most promising theories there was bound to be some risk. However Dr. Scott undertook to carry out the operation, and his account of the process, given in an addendum to this article (pp. 238-9), shows how completely he succeeded. From the scientific point of view it can hardly be denied that the dissemination of the knowledge of this chemical treatment of the leather is of greater value than the publication of the contents inscribed on it.

Glanville goes on to say that the hopes expressed by some (and particularly Eisenlohr) that the work was very important and perhaps even "the original of [the work on] the papyrus roll" were not realized by the actual contents. "In place of the hoped for treatise on Egyptian mathematics which was to explain all the difficulties in the Rhind Papyrus, we have a copy in duplicate of 26 sums in addition of fractions!"

But this pessimistic appraisal of the signficance of the leather roll for understanding the ancient Egyptian procedures involving fractions has been reversed as the result of the accounts of the leather roll given by Vogel, Neugebauer, van der Waerden and Gillings, as our account of Egyptian fractions in the fourth chapter has shown. The result is that the 26 equalities, twice given in the roll, are now considered highly relevant to our understanding of the Egyptian treatment of fractions.

The English translation given here as Document IV.5 is based on Glanville's translation, photographs of both copies of the hieratic text (Figs. IV.16a and IV.16b) and their hieroglyphic transliteration (Figs. IV.17a and IV.17b). It presents no significant paleographical problems. Columns 1 and 2 suffered from fragmentation and hence the intact copy in Columns 3 and 4 had to be used to restore the text of Columns 1 and 2. The restorations are indicated within closed brackets. That the second copy was made from the first seems clear from the fact that the trivial errors of the first copy reappear in the second copy.[5]

Notes to the Introduction to Document IV.5

[1] The introduction to this short document is rather abbreviated because it has been discussed at some length in Chapter Four.
[2] The initial description, edition, and translation of the roll was made by S.R.K. Glanville, "The Mathematical Leather Roll in the British Museum," *JEA*, Vol. 13 (1927), pp. 232-39.
[3] A. Scott and H.R. Hall, "Laboratory Notes: Egyptian Leather Roll of the 17th Century," *British Museum Quarterly*, Vol. 2 (1927), pp. 56-57, and 1 plate. See also Scott's "Addendum" to Glanville's article cited in note 2 above, pp. 238-39.
[4] Glanville, *op. cit.* in note 2 above, pp. 232-33.
[5] *Ibid.*, p. 233.

DOCUMENT IV.5

The Mathematical Leather Roll of the British Museum

[First Copy: Cols. 1 and 2; see Figs. IV.16a and IV.17a.]

[Col. 1:]	
[Lin. 1] 1/10 1/40	it is 1/8
[Lin. 2] 1/5 1/20	it is 1/4
[Lin. 3] 1/4 1/12	it is 1/3
[Lin. 4] [1/10] 1/10	it is 1/5
[Lin. 5] [1/6 1/6]	it is 1/3
[Lin. 6] [1/6 1/6 1/6]	it is 1/2
[Lin. 7] [1/3 1/3]	it is 2/3
[Lin. 8] [1/25] 1/15 [1/]75 1/200	it is 1/8
[Lin. 9] [1/50] 1/30 [1/]150 1/400	it is 1/16
[Lin. 10] [1/]25 [1/]50 1/150	it is 1/6 (*sic, should be* 1/15)[1]
[Lin. 11] [1/9] [1/18]	it is 1/6
[Lin. 12] [1/7] [1/14] 1/28	it is 1/4
[Lin. 13] [1/12] [1/2]4	it is [1/8]
[Lin. 14] 1/14 1/21 1/42	it is [1/7]
[Lin. 15] [1/18] [1/2]7 1/54	it is [1/9]
[Lin. 16] [1/12 (*sic?, should be* 1/22)[2] 1/33] 1/66	it is [1/11]
[Lin. 17] [1/28 1/49] 1/196	it is [1/13 (*sic?*, 1/14)[3]]
[Col. 2:]	
[Lin. 1] 1/30 1/45 1/90	it is 1/15
[Lin. 2] 1/24 1/48	it is 1/16

[Lin. 3] 1/18 1/36	it is 1/12
[Lin. 4] 1/21 1/42	<it is>[4] 1/14
[Lin. 5] 1/45 1/90	it is 1/30
[Lin. 6] 1/30 1/60	it is 1/20
[Lin. 7] 1/15 1/30	it is 1/10
[Lin. 8] 1/48 1/96	it is 1/32
[Lin. 9] 1/96 1/192	<it is>1/64

[Second copy: Columns 3 and 4; see Figs. IV.16b and IV.17b]

[Col. 3:]

[Lin. 1] 1/10 1/40	it is 1/8
[Lin. 2] 1/5 1/20	it is 1/4
[Lin. 3] 1/4 1/12	it is 1/3
[Lin. 4] 1/10 1/10	it is 1/5
[Lin. 5] 1/6 1/6	it is 1/3
[Lin. 6] 1/6 1/6 1/6	it is 1/2
[Lin. 7] 1/3 1/3	it is 2/3
[Lin. 8] 1/25 1/15 1/75 1/200	it is 1/8
[Lin. 9] 1/50 1/30 1/150 1/400	it is 1/16
[Lin. 10] 1/25 1/50 1/150	it is 1/6 (*sic, should be* 1/15)
[Lin. 11] 1/9 1/18	it is 1/6
[Lin. 12] 1/7 1/14 1/28	it is 1/4
[Lin. 13] 1/12 1/24	it is 1/8
[Lin. 14] 1/14 1/21 1/42	it is 1/7
[Lin. 15] 1/18 1/27 1/54	it is 1/9
[Lin. 16] 1/12 (*sic, in fact* 1/ 22) 1/33 1/66	it is 1/11
[Lin. 17] 1/28 1/49 1/196[5]	it is 1/13 (*sic*, 1/14)
[Lin. 18] 1/30 1/45 1/90	it is 1/15
[Lin. 19] 1/2[4] 1/4[8]	it is [1/1]6[6]

[Col. 4:]

[Lin. 1] 1/18 [1]/36[7]	it is 1/12
[Lin. 2] 1/21 1/42	it is 1/14

[Lin. 3] 1/45 1/90	it is 1/30
[Lin. 4] 1/30 1/60	it is 1/20
[Lin. 5] 1/15 1/30	it is 1/10
[Lin. 6] 1/48 1/96	it is 1/32
[Lin. 7] 1/96 1/192	<it is>[8] 1/64.

Notes to Document IV.5

[1] The error appears to be that the scribe looked at the conclusion of the next line and mistakenly wrote it down here.

[2] We do not know whether the error is present, but it probably is, since it is in the copy in line 16 of Column 3.

[3] This error is in line 17 of Column 3 and so was probably also here in Column 1.

[4] Omitted in both copies here and in line 9.

[5] van der Waerden would correct the expression to 1/26 + 1/39 + 1/78 instead of making the correction of 1/13 to 1/14, which I have here suggested on the economical basis of making one change rather than three.

[6] The bracketed numbers in this line are given in copy 1.

[7] As in copy 1.

[8] Omitted in both copies; see note 4.

DOCUMENT IV.6

Sections G-I from Reisner Papyrus I: Introduction

Document IV.6 consists of Sections G-I, which I have taken from the integral group comprised by Sections G-K of the so-called Reisner Papyrus I of the Museum of Fine Arts in Boston (see Figs. IV.18a-IV.18j). The chosen sections are those that represent best the mathematical formulations followed by the accountants. The editor of the Reisner Papyri, William Kelly Simpson, describes the discovery and the character of the Reisner Papyrus I succinctly:[1]

> The document which forms the basis of this publication [i.e., Reisner Papyrus I] is an Egyptian account papyrus of the second reign of Dynasty 12. It was found during the excavations conducted [in 1901-04] by the late Dr. George Andrew Reisner on behalf of the University of California at Nag^{c} ed Deir [the necropolis for the ancient town of This], a site roughly opposite Girgeh in Upper Egypt ...The papyrus is now in the Museum of Fine Arts [in Boston]....[and] bears the museum accession number 38.2062.
>
> The Papyrus was one of four rolls discovered lying on one of the three wooden coffins in tomb N 408..., as announced briefly by Reisner in 1904. A number of years after their discovery the rolls were sent to the late Dr. Hugo Ibscher, then the director of the Papyrus Sammlung of the Akademie der Wissenschaften in Berlin. Under his meticulous care the papyri were unrolled and the fragments put in place....
>
> The [Reisner]Papyrus [I]...measures approximately 3.50 meters long in its present state...has a maximum height of 31.6 cm. and is made up of nine sheets of papyrus, the first and last of which are incomplete and the remaining seven of which are approximately 42 cm. long....

> In common with many of the account papyri of the Middle Kingdom and the Old Kingdom, there are ruled guidelines to assist the scribe in aligning his writing.

Simpson goes on to discuss at some length the question of the dating of Reisner Papyrus I, and initially decides that it belongs to either the first or second reign of Dynasty 12 (those of Ammenemes I and Sesostris I). But his ultimate preference is for the reign of Sesostris I.[2]

As the title of my document indicates, I have confined it to the calculations of Sections G-I of the papyrus, but it will be useful to discuss briefly the whole group of Sections G-K. They consist of accounts that refer to a building construction. The section letters are ones assigned by the editor for the purposes of editing and reference. The year to which they apply is not given; nor is the reigning Pharaoh's name mentioned. But the editor suggests assigning them to the period between IV Peret 6 and II Shemu 20 of the year 24 of Sesostris I.[3]

As Simpson suggests,[4] the building for which the Sections G-K provide four accounts (G-J) and a summary of those accounts (K) appears to have been a temple, a cenotaph, or a tomb "since two of its component parts are designated as the august chamber and the eastern chapel....The final result is a record of the total number of man-days expended on the undertaking during the period involved, a relatively substantial figure of 4312 1/2 man-days. The general bearing of this type of calculation on the daily rosters and lists of men elsewhere in the document and in the other Reisner papyri is obvious, since one type records the details of the expenditure of labor and the other type provides the names of the individuals so employed." Being actual accounts these sections deal with completed, actual quantified activities, and thus differ, for the most part, from the preceding documents of this chapter, which are concerned largely with methods of calculating model practical arithmetical or arithmetical-geometrical problems that would involve whole numbers and fractions, or with tabulated calculations useful as reference tables for simplifying the Egyptian arithmetical procedures of doubling, halving, multiplying or dividing by ten, and summing or subtracting whole and fractional quantities. Since the principal object of the set of documents for Chapter Four is to illustrate the nature of Egyptian mathematics and its techniques, most of the documents included in this volume do just that. Hence the problems in those documents were in general contrived with simplified numbers and answers to avoid extensive series of calculations. But Document IV.6, as an illustration of an actual account rather than a didactic device, is the only lengthy one I have included to illustrate the techniques of account books in the present volume. However I have also included a Middle Kingdom table from the temple at Illahun that illustrates actual accounts and (like Document IV.6) shows the necessary procedure of approximating fractions in order that they may be realistically measured (see Figs. IV.33 and IV.34, as well as the section "Pefsu Problems" in Chapter Four above). Of course, the reader may also remember that I have frequently mentioned account books and tables of actual quantitative data in Volume Two, as for example, the day books of the Temple of Illahun (see the index of that volume under "Illahun").

Simpson's summary of the general content of the sections that make up this document is well worth quoting, along with his remarks about the complexity of the accounts:[5]

> Section G is an account of the *ḥmʿw* (*ḥmw?*), a term possibly to be understood as rubble or a type of earth, prepared, utilized, or removed in the construction of the building. Section H is a list, with measurements, of the blocks of stone hauled from the storehouse and the amount of sand carried for use in specific subdivisions of the operation. Section I deals with calculations involved in the various stages of brickwork and related materials. In Section J a detailed accounting is rendered for the distribution of manpower to the general work force, to specific subdivisions of it, and to miscellaneous tasks relating to the transport of

> personnel and materials, in which several varieties of wood figure prominently. The summary in Section K consists of a recapitulation of the expenditure of man-days in each of these operations. One cannot fail to be amazed and bewildered by the complexity of the bookkeeping practices whereby the accountant recorded such details as the exact dimensions of blocks of stone to a fraction of a fingerbreadth and the sum of man-days to a fifth of a working day. Even more extraordinary, if our present interpretation is correct, is the implied existence of a set of rigidly fixed ratios between the volume of the material operated on and the number of man-days expended or allowable for completing the specific tasks with this cubic content.

As a matter of fact, it may be remembered by the reader of the earlier volumes of my work that some similar rules seem to exist concerning the use of fractions in specifying the annual maximum height of the Nile in the yearly boxes of the Early Annals on Stone (=the so-called Palermo Stone), which I presented as Document I.1 (see Volume One, pp. 109-113), and in giving fractions of hours in a table of the lengths of daylight and nighttime in a late document of Dynasty 26 (see Volume Two, pp. 101-06).

The calculation of the accounts in the current document is, in principle, not of great difficulty since it involves the determination each day of the number of enlistees required for a given task, by (1) first multiplying the product of the listed length, width, and thickness or depth of the materials involved (i.e., the volume of the materials) by a specified number of like construction units demanding the same volumetric totals (usually 1 or 2 but once 4). This final volumetric total for each line of entry is then silently divided by 10, presumably indicative of the value of 10 cubic cubits that each enlistee would be able to do in a day. The final result, then, in the last column was the total number of enlistees or workers needed for that day's work, i.e., the allowed number of man-days for that day's task. The unit of linear measurement for the dimensions in these accounts is the cubit and/or its fractions (either expressed as some general unit fraction or sum of unit fractions or in terms of the palm equal to 1/7 of a cubit and the fingerbreadth equal to 1/4 palm). These are measures we have already discussed in Chapter Four and in the various documents that have preceded this one. One peculiarity is that although the volumetric answer is obviously in cubic cubits and its fractions, its fractional component is sometimes expressed in so many palms and or fingerbreadths, which terms are elsewhere confined to linear measures. It seems evident that in such cases what is meant by the terms is 1/7 and 1/28 of the cubic cubit.

Though the calculating formulas assumed in the Reisner Papyrus I are not themselves difficult and produced many wholly correct calculations, the fact that so much effort was exerted to specify the tabulated data in detailed fractions of a cubit as well in whole cubits tended to produce a number of errors, some serious and many merely slips. Still others were probably intentional approximations since only approximations of the fractional amounts were thought necessary. These errors and approximations I have pointed out in the course of Document IV.6 (leaving the correct calculations without comment). Both correct and erroneous entries have been retabulated in a series of tables by R. J. Gillings (see my Figs. IV.19 and IV.20) and discussed by him in a thoughtful but speculative and hardly conclusive way as follows:[6]

> My interpretation of Simpson's translation of Reisner Papyri I and II leads me to conclude that the chief overseer of the dockyard appointed a skilled scribe to instruct the tally clerk in multiplication and division for the records of the workshop. The scribe's teaching for integers and the simple fractions of a cubit like 2/3, 1/2, 1/3, 1/4 was well done, because of the 65 entries made by the clerk, there were 11 that contained only integral values of cubits, and for these the clerk's arithmetic is 100 percent accurate [e.g., see lines 15-16 of Section G]....

> In those lines where *[numerical] fractions* of cubits were included, only 3 errors occurred, so that in the 31 of these calculations, the clerk was 90 percent accurate. [For a correct example, see line 27 in Section H.]....
>
> There are 11 entries that include measures in cubits and palms but *no [entirely numerically expressed] fractions.* Now here the clerk began to find a little difficulty, because the palms must be expressed as fractions of a cubit, and so a table of fractions of a cubit would need to be prepared and handy for reference.

Gillings goes on to hypothesize such a table, namely, that in Fig. IV.21, which gives the successive values of 1 palm, 2 palms, 3 palms, 4 palms, 5 palms, and 6 palms as the sums of unit fractions of a cubit. But the calculation of a cubit volume where the dimensions included values in cubits and palms (with the latter converted to sums of unit fractions of cubits), for example, in calculating the cubic cubits of the volume in line 24 of Section H, would have involved the addition of some 33 fractions in total, which Gillings concluded was far too cumbersome. And so he believes that the scribe instead developed an easier method of calculating the volumes, which Gillings deduced from the manner in which line 11 of H is presented: [length:] 3; 1 palm [width:] 1 [depth:] 1 [volume:] 3; 1 palm:

> The volume, here found mentally, clearly means 3 cubic cubits and one seventh of a cubic cubit, although it is written as 3 cubits 1 palm. One seventh of a cubic cubit would be a flat rectangular prism, 1 cubit by 1 cubit by 1 palm, and from the analogy of the modern "superficial foot" in measuring timber, I will call this the Egyptian "superficial cubit." This I think is quite justified, since in every one of the RP [i.e., Reisner Papyrus] calculations, the volume is stated simply in cubits, palms, and fingers, without any suggestion

of square cubits or palms, nor of cubic cubits or palms. And this simplified method of stating the volume or cubic contents allows of a much easier way of doing the required multiplications, which I am sure the scribe invented and explained to the clerk, although I have no direct evidence to prove it, working only with what is to be deduced from the multiplications before us in the Reisner Papyri.

Gillings then takes the data of line 24 of Section H, i.e., length = 3c 5p, width = 1c 2 p, and depth = 6p, and using his suggested simplifying technique, he gets the following results: (3c 5p) x (1c 2p) x (6p) = 4c 0p and (4 4). The steps Gillings proposes to represent the Egyptian method of finding that result are shown in Fig. IV.22. Note that the (4 4) which is the last part of Gillings' determination of the volume with his newly suggested procedure differs from the 4c 2f found in the papyrus. That latter entry Gillings believes to be a scribal error, or at best a rough approximation of the (4 4). But I remain doubtful about the so-called "Egyptian superficial cubit" as a device to simplify the calculations with fractions. It seems to me that among the possible methods dealing with the extensive multiplication of fractions discussed by Gillings, his second method, which would reduce all the linear measures to palms and then divide the product (expressed in cubic palms) by 343 (which is the number of cubic palms in 1 cubic cubit), is more straight-forward and would obviously be the more familiar procedure to the Egyptian calculator, even though cumbersome. Of course, Gillings is perfectly correct in his conclusion that when fractions of volume are expressed as palms or as fingerbreadths in the final product, we must accept that they are used as volumetric fractions equal to 1/7 and 1/28 of a cubic cubit. Still, without the accountant's working notes we cannot be sure just what were his calculating procedures in the entries involving palms and fingers.

In my effort to check the accountant's determinations of the number of enlistees or man-days by dividing the volumes by 10, at least where fractions are involved, I have found useful the Egyptian

Table of the Divisions of the first 9 numbers by 10 given in Document IV.1 before Problem 1, as will be evident to the reader.

Document IV.6 follows Simpson's translation quite closely with only minor changes of form and explanatory additions.[7] I have also transcribed the seasons in the manner I followed in the earlier volumes: "Peret" and "Shemu" instead of Simpson's "Proyet" and "Shomu." For the hieratic texts and hieroglyphic transcriptions of all five Sections G-K (and not just those of Sections G, H, and I, which are given here as my Document IV.6), see Figs. IV.18a-j. As in the preceding documents, rubricated words are given in the translation in boldface instead of the more conventional underlining employed by Simpson. They are largely confined to the entries in the "units" column (e.g., see Section G, every line), to totals (G, 19; K, 17), to dates (H, 22,25,28,30-31,33-34; I, 1; J, 1, 10), to sundry other numbers and words (see underlined items, *passim*, in Simpson's translations [*op. cit*, pp. 125-28]), and to a title (K, 1). Note that I have added an extra space after each numbered line, though that was not my procedure in some of the previous documents. I have done this in this document to reduce the apparent clutter that would have resulted from squeezing them together. I remind the reader that the full edition of this and the other Reisner Papyri has been masterfully executed by Simpson and the reader will profit greatly by consulting that edition at first hand.

Notes to the Introduction to Document IV.6

[1] *Papyrus Reisner I: The Records of a Building Project in the Reign of Sesostris I. Transcription and Commentary* (Boston, 1963), pp. 17-19.
[2] *Ibid.*, pp. 19-21.
[3] *Ibid.*, p. 22.
[4] *Ibid.*, p. 52.
[5] *Ibid.*
[6] R. J. Gillings, *Mathematics in the Time of the Pharaohs*, pp. 223-31, covering this and the next three quotations and accompanying discussion. Note that

throughout Gillings' work he has used the reciprocal form of writing unit fractions, i.e., with the denominator number superscribed by a horizontal bar (and no unit denominator). For example, see Figs. IV.19-IV.21.

[7] *Op. cit.* in note 1, pp. 124-26.

DOCUMENT IV.6

Sections G-I from Reisner Papyrus I

[Section G; see Figs. IV.18a and IV.18b]

[Lin. 1][1] [............................][2] [Column heads:][3]
[a.] [Leng]th ([ꜣ]*w*) [b.] width (*wsḫ*) [c.] [thickness or depth] (*mḏwt*) [d.] <units> [e.] product or volume (*sty*) [f.] [the calculation of enlistees (*ỉr m ḥsb*, i.e., the needed number of enlistees for the day's work[4])].

[Lin. 2] [....................] [a.] 38 [b.] 12 [c.] [7]? [d.] [1] [e.] 3[192]? [f.] [319 1/5]?

[Lin. 3] [....................] [a.] 25 [b.] 20 [c.] [...] [d.] [...] [e.] [...] [f.] [...].

[Lin. 4] [......... the august chamber] [a.] 15 [b.] 5 [c.] [...] [d.] [...] [e.] [.]5 [f.] [...].

[Lin. 5] [............] tr. in this chamber [a.] 3 [b.] [2] [c.] [2] [d.] 1 [e.] 12 [f.] 1 [1/5].

[Lin. 6] [given to him in] the eastern chapel of the *akhty (?)* [a.] 8 [b.] 5 [c.] 1/4 [d.] 1 [e.] 10 [f.] 1.

[Lin. 7] portal of this chamber [a.] 3 [b.] 2 1/2 [c.] 1/4 [d.] 1 [e.] 2 1/2 *(sic; should be* 1 1/2 1/4 1/8*)* [f.] 1/5 1/20 *(correct for vol.* 2 1/2, *but should be* 1/10 1/20 1/40 1/80*)*.

[Lin. 8] [(month and day)] given to him in [..........] [a.] 35 [b.] 11 [c.] [1/2] [d.] [1] [e.] [1]92 1/2 [f.] 19 1/2 (*should be* 19 1/4).

[Lin. 9] [a.] 13 [b.] 11 [c.] 1 1/2 [d.] 1 [e.] [2]14 1/2 [f.] 21 1/2 *(an approximation for* 21 1/4 1/5*).*

[Lin. 10] . [a.] 52 [b.] 3 [c.] 1/4 [d.] 1 [e.] 39 [f.] 4 *(approx. for* 3.9 *in modern terms or in Egyptian terms for* 3 2/3 1/5 1/30, *the fractions as given in the Table of Division by 10 in Doc. IV.1, before Problem 1).*

[Lin. 11] . [a.] 32 [b.] 4 [c.] 1/2 [d.] 1 [e.] 85 *(sic, but should be* 64) [f.] 8 1/2 *(correct for the erroneous* 85, *but for the proper* 64 *it should be* 6.4 *in modern terms, or in Egyptian terms* 6 1/3 1/15, *the fractions as given for the division of 4 by 10 in the Table of Division by 10 quoted above).*

[Lin. 12] [given to him] in the portal of the [king's] *ḏrit*-chamber as *ḫmʿw*
. [a.] 3 1/2 [b.] 2 [c.] 2/3 [d.] 1 [e.] 4 2/3 [f.] 1/2 *(approx. for* 1/3 1/10 1/30, *the Egyptian way of expressing* 1/10 of 4 2/3*).*

[Lin. 13] [...] Shemu 25, given to him [as] *ḫmʿw* in the chamber for emptying *(or read* for drying*)* [a.] 10 1/2 [b.] 8 1/2 [c.] 1/3 (?) [d.] 1 [e.] 27 *(sic, but should be* 29 1/2 1/4*)* [f.] 2 1/2 1/5 *(corr. for* 27, *but should be* 2 2/3 1/5 1/20 1/30 1/40*).*

[Lin. 14] [gi]ven [to him] in this chamber [a.] 8 [b.] 3 [c.] [1/3] [d.] 1 [e.] 8 [f.] 1/2 1/4 1/20 *(which is obviously equivalent to* 2/3 1/10 1/30, *the value used in the Table of Division by 10 for 8 divided by 10).*

[Lin. 15] [...] Shemu 27, given to him in *ḫmʿw* to support (?) the column-bases (?) [a.] 6 [b.] 4 [c.] 2 [d.] 1 [e.] 48 [f.] 4 1/2 1/4

1/20 *(cf. the unit fractional expression for the division of 8 by 10 in the preceding line).*

[Lin. 16] [a.] 4 [b.] 2 [c.] 2 [d.] 1 [e.] 16 [f.] 1 [1/2] 1/10.

[Lin. 17] [(month and day)] given to him [........] "footings" [.......]

[a.] 4 [b.] 4 [c.] 2 [d.] 2 [e.] [64] [f.] 6 [1/4] 1/10 1/20.

[Lin. 18] [a.] 3 [b.] 3 [c.] 2 [d.] 2 [e.] [3]6 [f.] 3 1/2 1/10

[Lin. 19] [Totals] [e.] **4335** [f.] **435**

[Section H; see Figs. IV.18c and IV.18d]

[Notice that only the product/volume column has a heading in this section. The column letters employed by the editor are the same as in Section G and indeed the succeeding sections. The final column concerning enlistees is not present in this section. The linear measure without verbal identification is always the cubit. I followed the editor in using a semicolon to separate it from any number of palms which are given in the text. In fact I have followed the editor in spelling out "palms" and abbreviating fingerbreadths "fb." Gillings' suggested restorations of erroneous numbers given below are found in Fig. IV.20.]

[Lin. 1] I Shemu 16, given to him in (for?) hauling stone from the st[orehouse].

[Lin. 2] [a.] 2; 5 palms [b.] 6 palms [c.] [.....(restored by Gillings as 6 palms, but with the comment "probably too great")] [d.] 2 [e.] 4; 1 1/3 fb.

[Lin. 3] [a.] 2 1/4 [b.] 6 palms [c.] [..... (restored by Gillings as 5 palms)] [d.] 1... [e.] 1...; 2 palms; 2...fb. (calculated as 1c; 2p; 1/2 1/14 fb. if the value of [e.] as 5 palms is assumed).

[Lin. 4] [a.] 2 [1/4 (? suggested by Gillings)] [b.] tr. (= "trace" here and below) (6 palms) [c.] [.....(restored by Gillings as 5 palms)] [d.] [.... (1?)] [e.] [....] 3 palms; [...].

[Lin. 5] [a.] 2 1/4 [b.] 6 palms ... [c.] [..... (restored by Gillings as 5 palms 2 fb. with the comment "probably too great")] [d.] 1... [e.] 1; 3 palms; 1 1/2 fb.

[Lin. 6] [a.] 4 1/4 [b.] tr. (6 palms?) [c.] [.....] [d.] 2 [e.] 6 1/2 1/4.

[Lin. 7] [a.] 3 1/4 [b.] tr. (1 1/2) [c.] tr. (1/2) [d.] 2 [e.] 4 1/2 1/4 1/8.

[Lin. 8] [a.] 2 1/2 [b.] 1 1/2 [c.] 1/2 [d.] 1 [e.] 1 1/2 1/4 1/8.

[Lin. 9] [a.] 4 1/2 [b.] [..... (1 1/2; *suggested by Gillings*)] [c.] tr. 1 + **1/3** *(which fraction should be deleted)* [d.] 2 [e.] 13 1/2 [*if Gillings' b. is correct, then the pap. erronoeusly adds here* 6, 4, 1; 4 p.].

[Lin. 10] [a.] 4; 1 palm [b.] 1 1/2 [c.] 1/2 tr. [d.] 1 [e.] 3; 2 (! 3) fb.

[Lin. 11] [a.] 3; 1 palm [b.] 1 [c.] 1 + **1/3** *(which rubricated fraction should be deleted)* [d.] 1 [e.] 3; 1 palm.

[Lin. 11B] [d.] [Total:] 15 (*sic, should be* 14 ?)

[Lin. 12] I Shemu 28, *spt* (column-base ?)

[Lin. 13] [a.] 2; 3 palms [b.] 2; 3 palms [c.] 2/3 [d.] 1 [e.] 3; 6 palms; 1 1/3 fb. (*Gillings sees this as a minor error for* 3; 6 palms; 2 1/14 1/42 fb.).

[Lin. 14] I Shemu 29 [a.] 2; 2 fb. [b.] 1 1/2 [c.] 1; 1 palm; 1 fb. [d.] 1 [e.] 3; 4 palms; 1 1/2 fb. (*Gillings sees this as a minor error for* 3; 4 palms; 2 1/2 1/28 fb.).

[Lin. 15] [a.] 1; 5 palms [b.] 1 1/2 [c.] 5 palms [d.] 2 [e.] 3; 5 palms (*Gillings sees this as a minor error for* 3; 4 palms; 2 1/2 1/4 1/14 1/28 fb.).

[Lin. 16] [a.] 2; 3 palms [b.] 1; 4 palms [c.] 5 palms; 2 fb. [d.] 1 [e.] 2; 5 palms; 2 1/2 fb. (*Gillings sees this as a major error for the approximation* 2; 6 palms; 3 1/2 fb.).

[Lin. 17] [a.] 4 [b.] 1; 3 palms [c.] 1 [d.] 4 [e.] 22; 6 palms.

[Lin. 18] [a.] 3; 2 palms [b.] 1; 2 palms [c.] 6 palms [d.] 1 [e.] 3; 3 palms; 2 1/3 fb. (*Gillings sees this as a minor error for* 3; 4 palms; 1 1/3 1/15 fb.).

[Lin. 19] [a.] 3; 5 palms; 2 fb. [b.] 1; 3 palms... [c.] 1 [d.] 1 [e.] 4; 2 palms; 3 fb. (*Gillings sees this as a major error for* 5; 2 palms; 3 1/4 1/7 1/28 fb.).

[Lin. 20] [a.] 3; 3 palms [b.] 1; 3 palms [c.] 1 [d.] 1 [e.] 5 (*Gillings sees this as an approximation or error for* 4; 6 1/4 1/28 palms).

[Lin. 21] [a.] 1; 5 palms [b.] 1; 3 palms [c.] 1 [d.] 1 [e.] 2; 4 palms; 1 ... fb. (*Gillings sees this as a major error for* 2; 3 palms; 1/2 1/14 fb.).

[Lin. 22] [a.] 1 2/3 [b.] 1; 3 palms [c.] 1 **-I Shemu last day-** [d.] 1 [e.] 2; 2 (?) palms; 2 ... fb. (*Gillings sees this as a minor error for* 2; 2 palms; 2 2/3 fb.).

[Lin. 23] [a.] 4 [b.] 1; 6 palms [c.] 6 palms [d.] 1 [e.] 4; 1 fb. (*sic; Gillings sees this as a major error for* 6; 2 palms; 2 1/4 1/28 fb.).

[Lin. 24] [a.] 3; 5 palms [b.] 1; 2 palms [c.] 6 palms [d.] 1 [e.] 4; 2 fb. (*Gillings sees this as a minor error for approximately* 4; 2 1/3 1/5 1/15 fb.).

[Lin. 25] [a.] 1 1/2 [b.] 1/2 1/4 [c.] 1 **-II Shemu 1-** [d.] 2 [e.] 2 1/4. Total: 100; 5 palms; 3 fb.

[Lin. 26] [a.] 2 1/2 [b.] 1/2 1/4 [c.] 1 [d.] 2 [e.] 3 1/2 1/4.

[Lin. 27] [a.] 3 1/2 [b.] 1 1/2 [c.] 1 1/2 [d.] 2 [e.] 15 1/2 1/4.

[Lin. 28] [a.] 4; 4 palms [b.] 1 ...; 5 palms [c.] 1; 2 palms **-II Shemu 7-** [d.] 2 [e.] 20; 1 palm; 1 1/2 fb. 23 (*sic; Gillings has the scribe's value as* 20; 1 palm; 1 1/2 fb., *and sees this as a minor error for approximately* 20; 1 palm; 1/4 fb.).

[Lin. 29] given to him as fill (*ḏbꜣw*) consisting of sand: the great chamber.

[Lin. 30] [a.] 12 [b.] 5 [c.] 1/4 **-IIII Peret-** [d.] 1 [e.] 15.

[Lin. 31] [a.] 15 [b.] 5 [c.] 1/4 **-day 15-** [d.] 1 [e.] 18 1/2 1/4.

[Lin. 32] eastern chapel [a.] 8 [b.] 5 [c.] 1/4 **to** [d.] 1 [e.] 10.

[Lin. 33] "footings" [a.] 4 [b.] 4 [c.] 2 **-I Shemu-** [d.] 2 [e.] 64.

[Lin. 34] [a.] 3 [b.] 3 [c.] 2 **-day 26-** [d.] 2 [e.] 36.

[Lin. 35] loads of the *Per-Sa* [a.] 8 [b.] 5 (*sic, but should be* 9) [c.] 1/4 [d.] 1 [e.] 18.

[Lin. 36] 50 (?)... rest.

[Lin. 37] great portal [e.] [total:] 100 1/2 69 (?) 45 1/2.

[Lin. 38] chapel [e.] 16.

[Lin. 39] *spt* (column-base?) [e.] 3; 6 palms; 1 fb.

[Lin. 40] great chamber [e.] 23 1/2 total: 143; 2 palms.

[Lin. 41] sand [e.] 143 1/2 1/4.

[Section I; see Figs. IV.18e and IV.18f]

[The column heads [a.]-[e.] stand for the same measures as in Sections G and H.]

[Lin. 1] **IIII Peret 15.**

[Lin. 2] given [to him] in (for?) striking ground: the great chamber. [a.] 12 [b.] 5 [c.] 1/2 [d.] 1 [e.] 30.

[Lin. 3] [given to him in] the august chamber [a.] [1]5 [b.] 5 [c.] 1/2 [d.] 1 [e.] 37 1/2.

[Lin. 4] *ḫ3* [(month and day)] given to him in the eastern chapel of the *akhty* (?). [a.] 8 [b.] 5 [c.] [1/2] [d.] 1 [e.] 20.

[Lin. 5] [(month and day) gi]ven to him in ... for..... [a.] [1]8 [b.] 1[1] [c.] [2/3?] [d.] 1 [e.] 132.

[Lin. 6] the western *mḫ3w* [a.] 32 [b.] 4 [c.] [1/4] [d.] 1 [e.] 32.

[Lin. 7] *ḫ3* the eastern *mḫ3w* [a.] 52 [b.] 3 **298...** ***(delete)*** [c.] [1/4] [d.] 1 [e.] 39.

[Lin. 8] [I (?) Shemu] 28, given to him as fill (*ḏb3w*): the great chamber [a.] 24 [b.] 5 palms [c.] 1/2 [d.] 1 [e.] 8; 4 palms.

[Lin. 9] [given to] him in (for?) carrying *srft* [a.] 26 [b.] 6 [c.] 5 palms [d.] 1 [e.] 111; 3 palms.

[Lin. 10] *tr*. [a.] 20 [b.] 5 [c.] 5 palms [d.] 1 [e.] 71; 3 palms.

[Lin. 11]

[Lin. 12] given to him in (for?) loosening brick-clay [a.] 27 [b.] 7 [c.] 2 [d.] 1 [e.] 378.

[Lin. 13] II Shemu 1, given to him in (for?) removing (?) water from the field [a.] 8 [b.] 7 [c.] 2 [d.] 1 [e.] 112.

[Lin. 14] II Shemu 2, given to him with the builders in the tower [a.] 1 1/2 [b.] 1 1/2 [c.] 2 [d.] 2 [e.] 9 [total:] **380.**

[Lin. 15] . [a.] 2 1/2 [b.] 1 1/2 [c.] 1 1/2 [d.] 2 [e.] 11 1/4.

[Lin. 16] . [a.] 3 1/2 [b.] 2 1/2 [c.] 1 1/2 [d.] 2 [e.] 25 1/4 (*sic, should be* 26 1/4).

[Lin. 17] . [a.] 4 [b.] 2 1/2 **81 1/2** *(delete bold)* [c.] 1 1/2 [d.] 2 [e.] 36 (*sic, should be* 30).

[Lin. 18] II Shemu 11, completed for him in brick-clay of the fields [a.] 10 [b.] 5 1/2; **6 palms** *(delete bold)* [c.] [?] 1/4 (*should be* 1) [d.] 1 [e.] 55.

[Lin. 19] [a.] 16 [b.] 5 1/2 ..... [c.] [?]; 6 palms [d.] 1 [e.] 75 1/4 1/5.

[Lin. 20] [a.] 8 [b.] 6 **5[56 1/4]** *(delete bold)* [c.] 1 [d.] 1 [e.] 48.

[Lin. 21] completed for him in large-size brick.

Notes to Document IV.6

[1] The line numbers are those given in Fig. IV.18a.
[2] The headings written in lacuna here in Section G and elsewhere in Section I probably had some reference to the month, the season, and the day, according to the evidence in lines 13 and 15 in G, lines 1, 12, 14, 22, 25, 28, 30-31, 33-34 in H, and lines 1, 8, 13, 14, and 18 in I, and many of the lines in J. As I noted in the introduction to this document, Simpson has suggested that the year of these accounts may have been the 24th year of the reign of Sesostris I. There are occasional references in the general line headings to the part of the building where the work is being done (e.g., lines 4-7, 12-15 of G, lines 1, 12, 29, 32; 37-40 of H, and *passim* in Section J).
[3] The letters assigned to the column heads in brackets are those assigned by the editor, and I have retained them throughout the document.
[4] For a discussion of this column head, which is restored here from lines I-1 and J-1, see Simpson, *op. cit.*, p. 53: "[Its form is] to be considered as a subjectless

passive ...with the sense, 'it is made as [so many] enlistees,' the pertinent number of enlistees recorded in each line under the heading. Now it is a curious fact that the figure for the number of enlistees, more correctly the number of man-days, is one tenth of the corresponding figure in the same line for cubic content in [column e of] the same line. The only obvious interpretation that occurs to me is that one man, working a full day, averages the preparation, removal, or putting into place of ten cubic cubits of the material known as *ḫmꜥw* [rubble?]." For a longer discussion of the volumetric calculations and the ratio of volume to man-days, see pp. 83-85 of Simpson's work. For *ḥsb* as "workman" see R.O. Faulkner, *A Concise Dictionary of Middle Egyptian* (Oxford, 1962, repr. 1972), p. 178.

Part Three

Bibliography and Indexes

Abbreviations Used in Text and Bibliography

ASAE = *Annales du Service des Antiquités de l'Égypte*, Cairo.
BMLR = The Mathematical Leather Roll in the British Museum.
BP = Berlin Papyrus 6619.
CRMP = Chace, A.B., et al., *The Rhind Mathematical Papyrus*, 2 vols. (Oberlin, Ohio, 1927, 1929). See fuller reference in Bibliography.
JEA = *The Journal of Egyptian Archaeology*.
KMP = The Kahun Mathematical Papyri.
MMP = The Moscow Mathematical Papyrus.
PSBA = *Proceedings of the Society of Biblical Archaeology*.
PRMP = Peet, T.E., *The Rhind Mathematical Papyrus* (London, 1923). See Bibliography.
QSGM = *Quellen und Studien zur Geschichte der Mathematik*.
QSGMAP = *Quellen und Studien zur Geschichte der Mathematik, Astronomie, und Physik*.
RMP = The Rhind Mathematical Papyrus.
RPI = The Reisner Papyrus I of the Museum of Fine Arts in Boston.
Wb = See the Bibliography, Erman, A., and H. Grapow.
ZÄS = *Zeitschrift für ägyptische Sprache und Altertumskunde*.

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Index of Egyptian Words and Phrases

The Egyptian words of this index are given in True Type, a transliteration font designed to run under Word for Windows. It was created as part of *Glyph for Windows* (Utrecht, Paris, 1993) which was programmed by Hans van den Berg and is used throughout my third volume. When a glyph or glyphs of the indexed item appears, the letter "g" has been added to the page number. If brackets are added to the transliteration, this indicates that only the glyph is present on the indicated page. Words beginning with either of the two signs for "s" (i.e., 𓊃 and 𓋴) are included under "s" in the index.

The reader may also wish to consult the glossaries or indexes found in Peet's edition of the Rhind Papyrus, Chace's edition of the same papyrus, Struve's edition of the Moscow Papyrus, and Couchoud's treatment of Egyptian mathematics, the titles of which are given in the bibliography above. The reader will also find many other Egyptian words given in these glossaries that are not in my index. For the glyphs of the powers of ten, see page 2, and for the signs of the Horus-eye fractions of a *heqat*, see Fig. IV.3 and pp. 66-67, 98 n. 13, and 192 n. 46, 224. The section on numbers and measures in Lesson XX (pp. 191-200) of Gardiner's *Grammar* (see Bibliography) is also useful. Needless to say, the great *Wörterbuch* of Erman and Grapow (see Bibliography) will be indispensable to the reader seeking other references to these words. The abridged *Ägyptisches Hand-wörterbuch* (Berlin, 1921; reprinted at Hildesheim, 1974) of these authors is helpful. Also useful is R.O. Faulkner's *A Concise Dictionary of Middle Egyptian* (Oxford, 1962; reprinted 1972).

Finally, the reader will readily see that the Index of Proper Names and Subjects below contains Egyptian words that have been transliterated with the addition of the letter "e" between consonants for the purpose of making them pronounceable.

šꜥty ("shaty," i.e., the value and weight of 1/12 deben): 169; see also the discussion of shaty and deben on p. 201, note 92.

šš ("grain"): 154.

šsp (a "palm," i.e., the sign is a hand without thumb = 1 palm = 4 digits): 8g. (two such signs, one above the other, is the measure of two palms; see p. 8g); *šspt:* 228 n. 4g.

ḳ

ḳnbt ("right angle" or "corner"–here only the determinative  appears and is the sign for square root): 49, 230 n. 10, 245.

ḳy n ššmtf ("method of reckoning it"): 156.

g

gm ("find"): 127.

t

tꜣ ("land" or strip of land 100 cubits x 1 cubit, i.e., 100 sq. cub.; hence a measure of area called a "cubit-strip," a "cubit area" or "a hundred [of land]"): 12, 99 n. 14.

twnw ("excess" or "difference"): 155.

tp ("example"): 134, 136, 139, 146, 155, 186 n. 14, 247 n. 5; for a discussion of its use in title of Rhind Papyrus and elsewhere, see note 1, pp. 183-84g.

tpr or *tp-r* ("base" of a triangle; lit. "mouth"?): 70, 232-33 n. 18.

ṯ

ṯnwt ("counting"): 3.

d

dbḥ ("a vessel used in measuring grain"): 180.

dmḏ ("total"): 186, n. 8.

ds ("a jug," a measure of beer): 15g.

ḏ

ḏꜣt ("remainder"): 101 n. 27, 186 n. 9.

Index of Proper Names and Subjects

My indexing of proper names is quite complete. Less complete is the indexing of geometric and other common terms that occur throughout the volume, i.e., words like square, triangle, fraction, quantity, etc. In such cases I have singled out the chief sections in the volume that consider problems involving the nature of, or the techniques and formulas for determining, such entities. The reader will find a list in the Table of Contents of the subjects treated in Chapter Four. Since that chapter attempts a survey and analysis of ancient Egyptian mathematics as a whole, this list of section heads is a ready guide to the most important subjects in the volume.

Though in the preceding index I have given, in their accepted form of transliteration, the Egyptian words mentioned in the volume, here I have added references to their oft-used pronounceable forms. The reader should therefore consult both indexes for any given term, e.g., *ḥḳȝt* in the first index and "heqat" in this index. The reader is reminded that, as in the case of Volume Two, very often there is more than one instance of the indexed word on the page cited.

The Bibliography is not indexed here since it is a single alphabetical list of the works used and cited. But the names of the authors mentioned in the text and endnotes are of course indexed.

A.

C.

D.

E.

Part Four

Illustrations

Introduction to the Illustrations

The illustrations which occupy the next 145 pages have, for the most part, been taken from the chief publications in which appear editions, transcriptions, or evaluations of the sources that I have used to present Egyptian mathematics in this volume. While many of them are used to illustrate or supplement my own evaluations of the principal doctrines and problems, and are of course tied by figure numbers to the texts and notes, some of them present the hieratic texts and hieroglyphic transcriptions of the six main documents. These latter figures are all mentioned by their proper numbers in the introductions to the documents, so that a reader of a given document can flip to the correct figure or figures. However, following a suggestion of one of my reviewers, I thought it might also be useful to specify here the numbers of the figures that carry the text of each of the documents:

Document IV.1: Figs. IV.2a-aaa.
Document IV.2: Figs. IV.6a-t.
Document IV.3: Figs. IV.12.
Document IV.4: Figs. IV.13a-b, IV.14a-b, and IV.15a-b.
Document IV.5: Figs. IV.16a-b and IV.17a-b.
Document IV.6: Figs. IV.18a-j.

Sometimes the sources quoted in the captions to the figures appear in a somewhat abbreviated form. But the reader can easily find the full biographical citations of those sources in the Bibliography at the beginning of Part III of the volume.

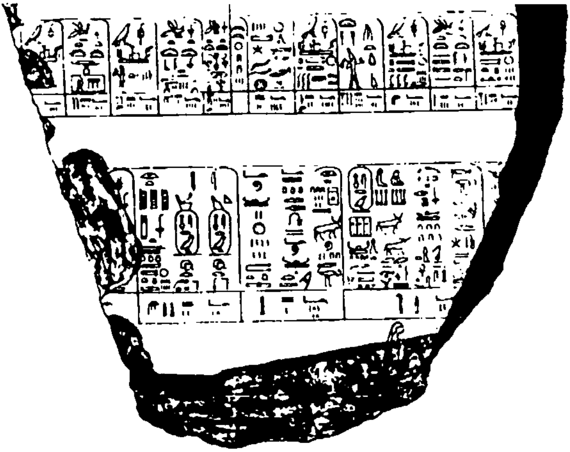


Fig. IV.0 Lines 5 and 6 of the recto of the Palermo Stone as redrawn by Pellegrini and extracted from Fig. I.33 of Vol. I of my *Ancient Egyptian Science* (cf. the photo of Fig. I.32). The lines are divided into year-boxes (read from right to left). The separate section at the bottom of each box gives the reading of the Nile height for the year of the box. In line 5, boxes 1 and 4, we see at the left end of the Nile section the sign for the fraction 1/2: ; in box 3 of that line, the sign for 2/3: ; and in line 6, box 4, the sign for 3/4: . Note that Gardener in his *Grammar*, p. 452, gives the sign for 2/3 as and that of 3/4 as .

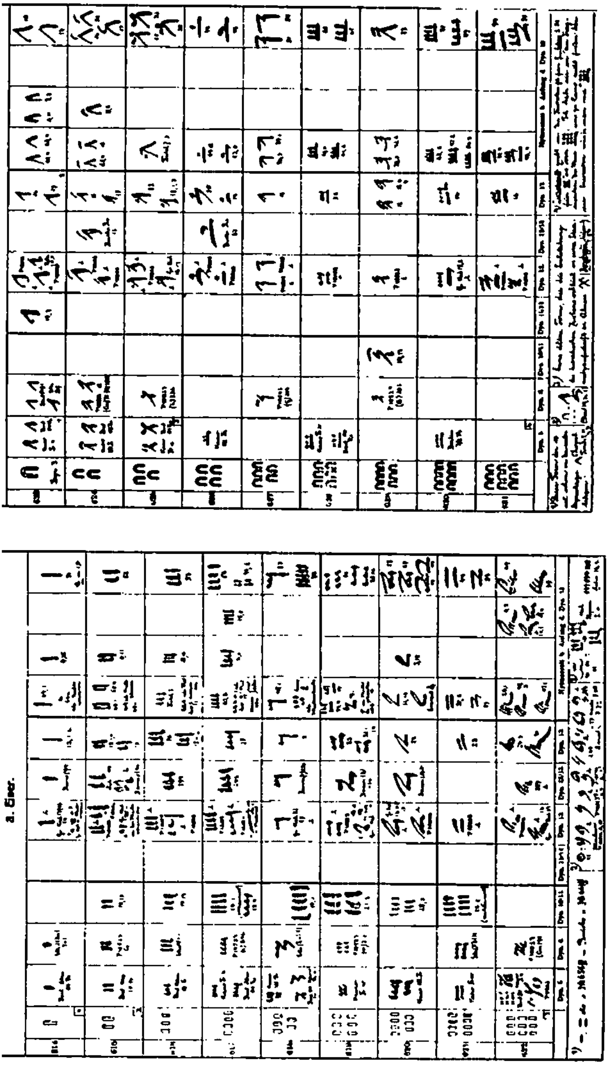


Fig. IV.1a Hieratic numbers (units and tens) up to the beginning of the 18th dynasty. Taken from G. Möller. *Hieratische Paläographie*, Vol. 1, pp. 59-60.

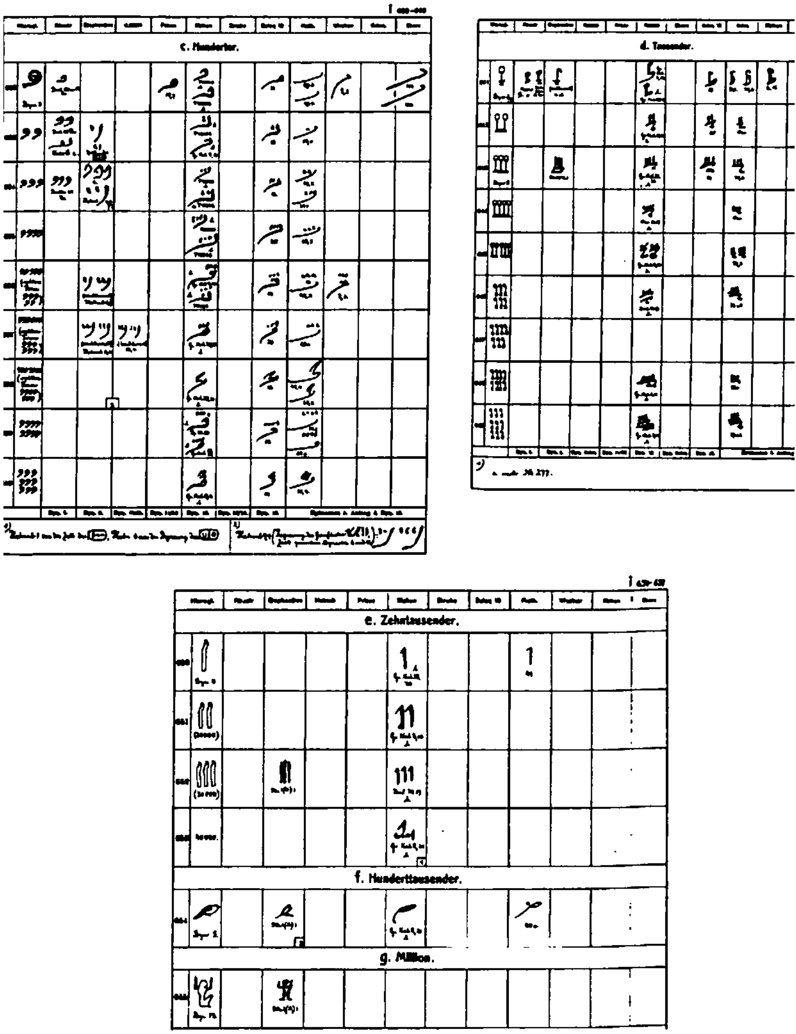
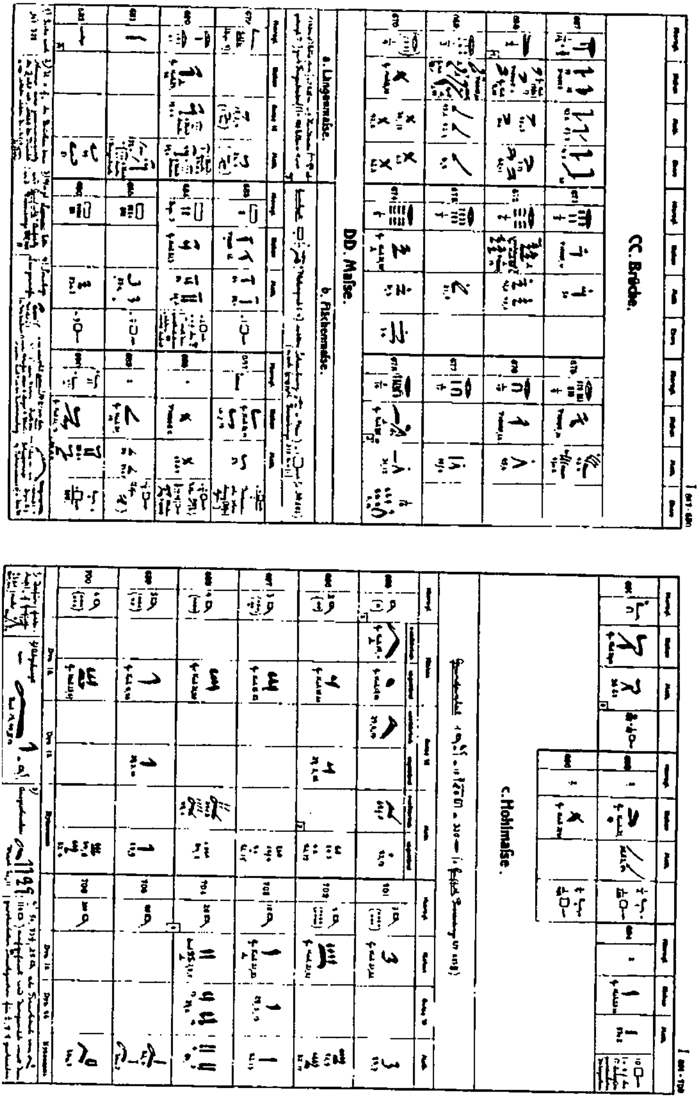


Fig. IV.1b Other powers of ten, *ibid.*, pp. 61-63.



CC. Brüche.
DD. Maße.
a. Längenmaße.
b. Flächenmaße.
c. Hohlmaße.

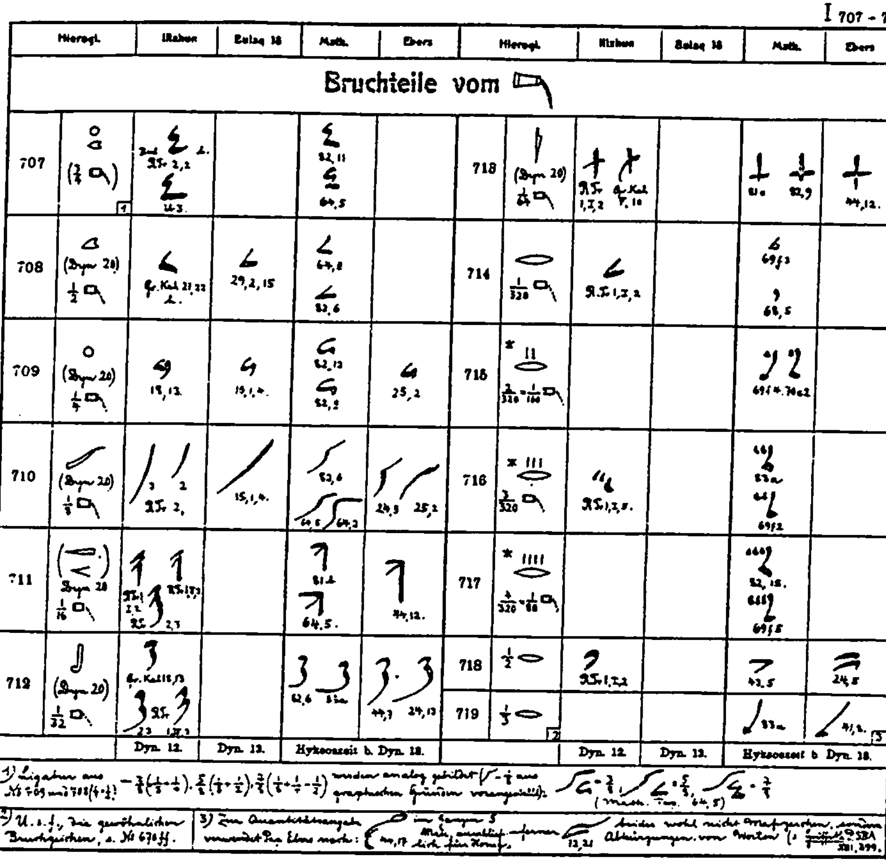


Fig. IV.1d Horus-eye fractions of a heqat, *ibid.*, p. 67.

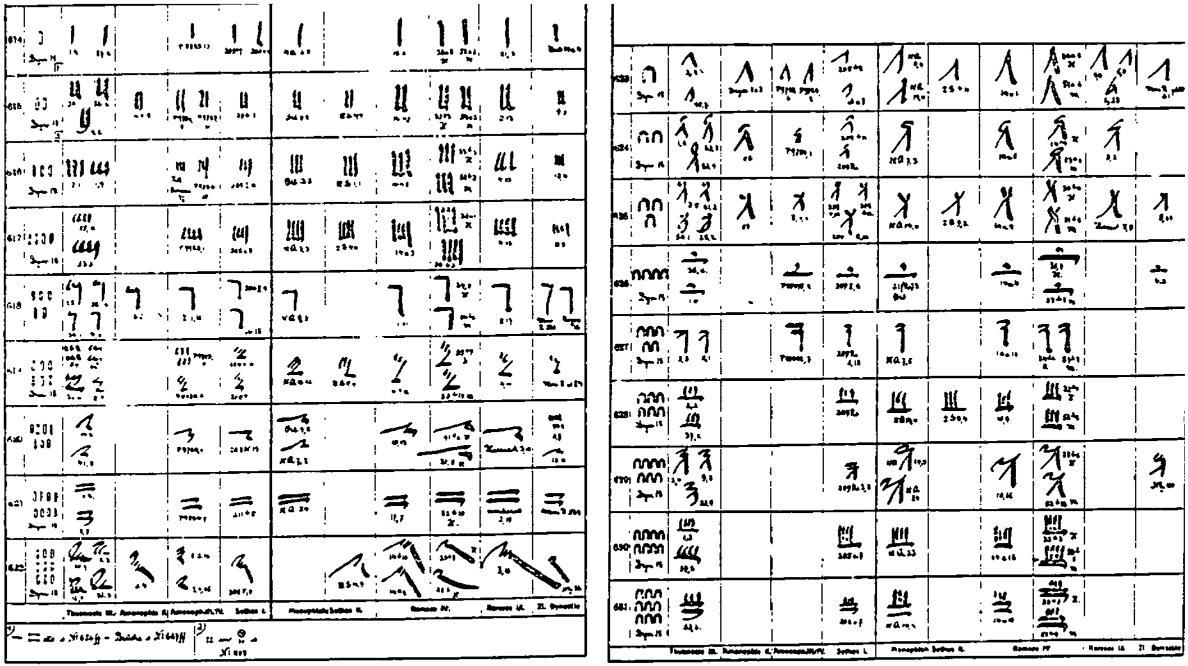
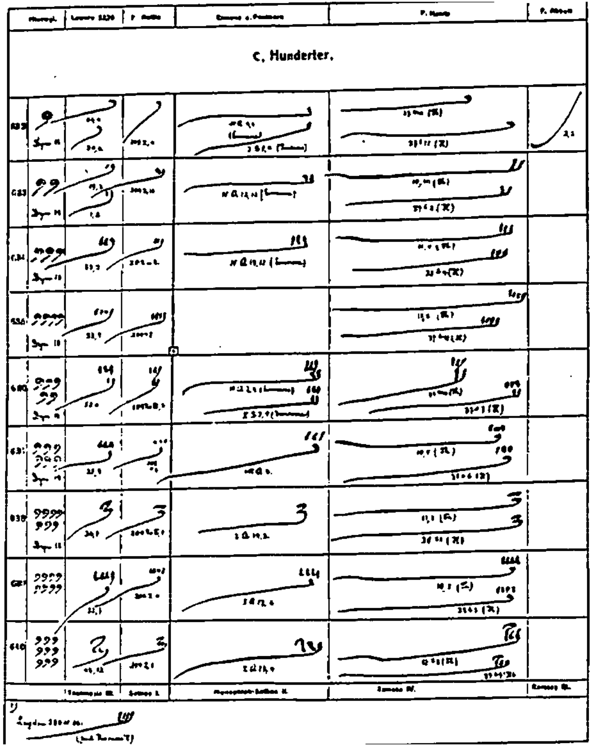
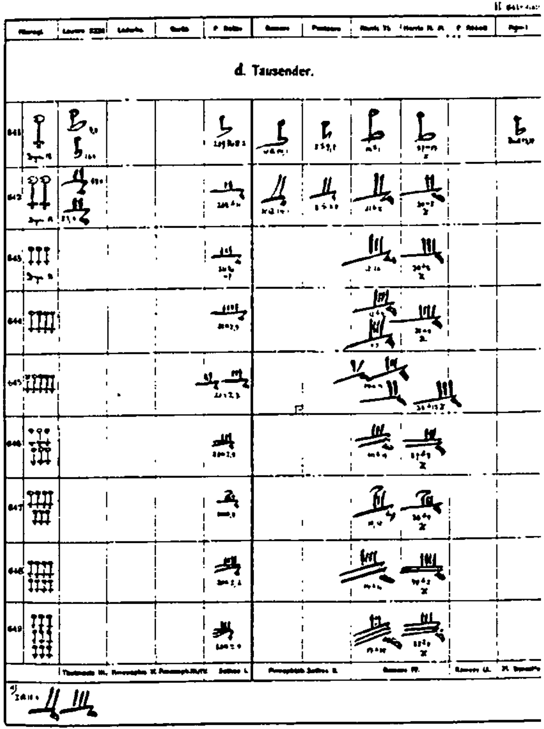
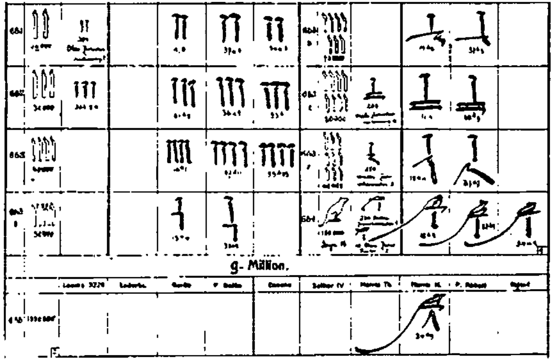


Fig. IV.1e Hieratic numbers (units and tens) from the time of Thutmosis III up to the end of the 21st dynasty. Taken from *ibid*. Vol 2, pp 55-56


c, Hunderter.


d. Tausender.



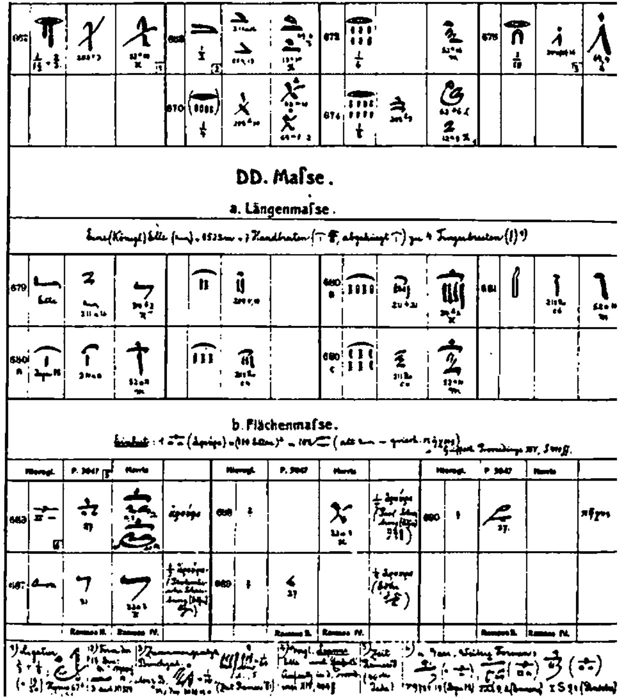


Fig. IV.1g 10,000s, 100,000s, and millions; fractions; length and area measures, *ibid.*, pp. 59, 61

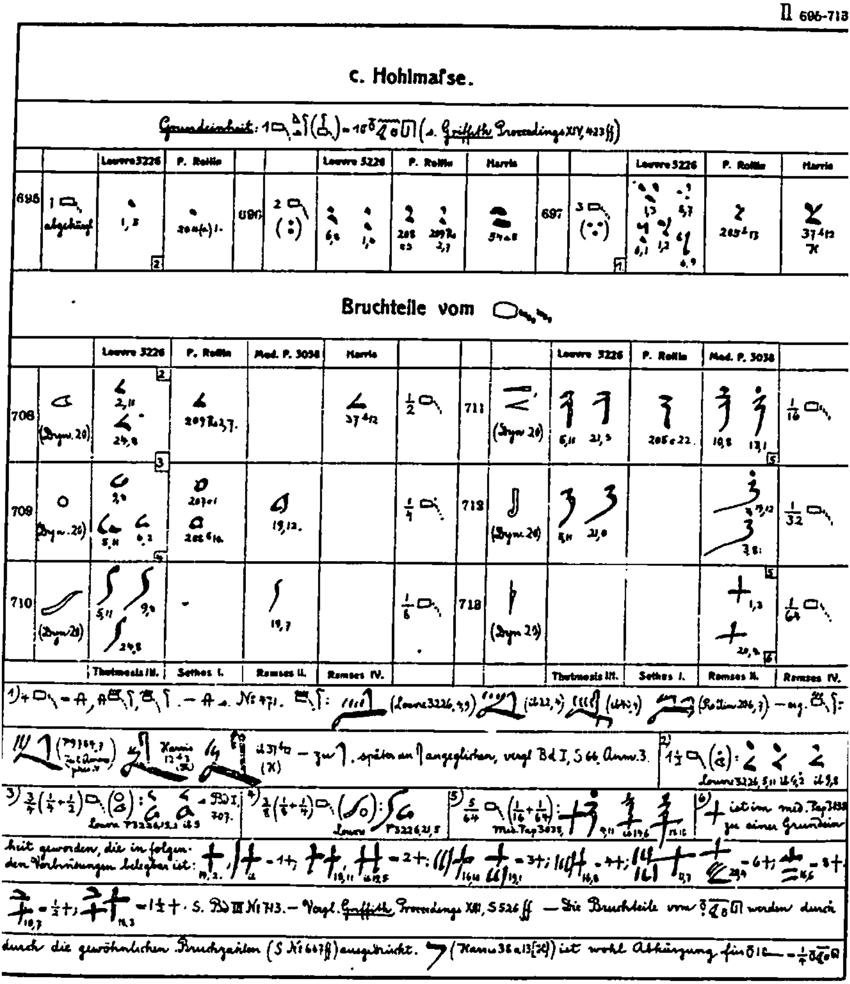
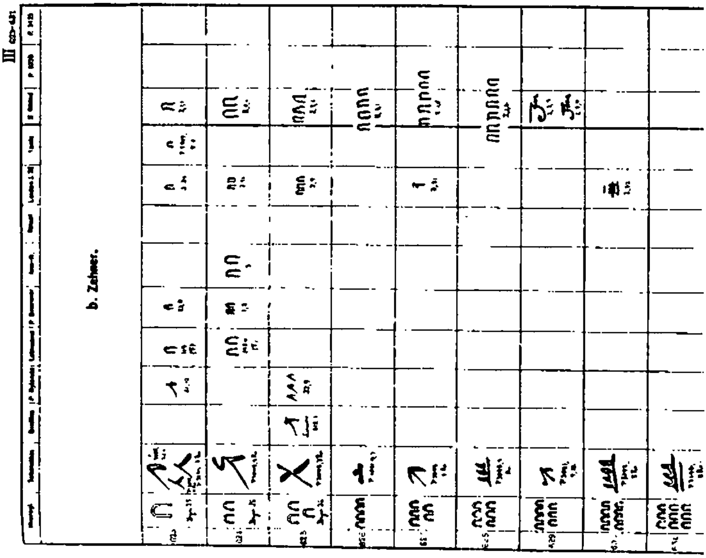
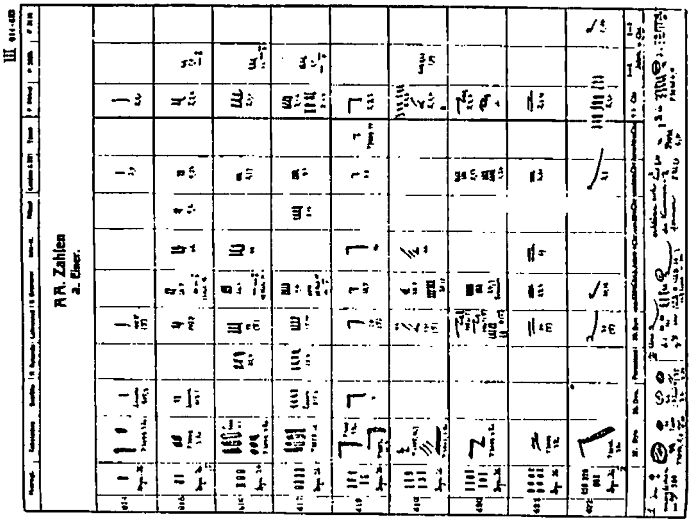
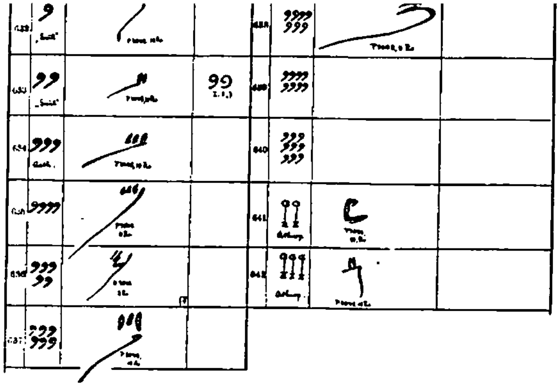


Fig. IV.1h Dry measures and Horus-eye fractions of a heqat, *ibid.*, 62.


b. Zehner.


AA. Zahlen
a. Einer.



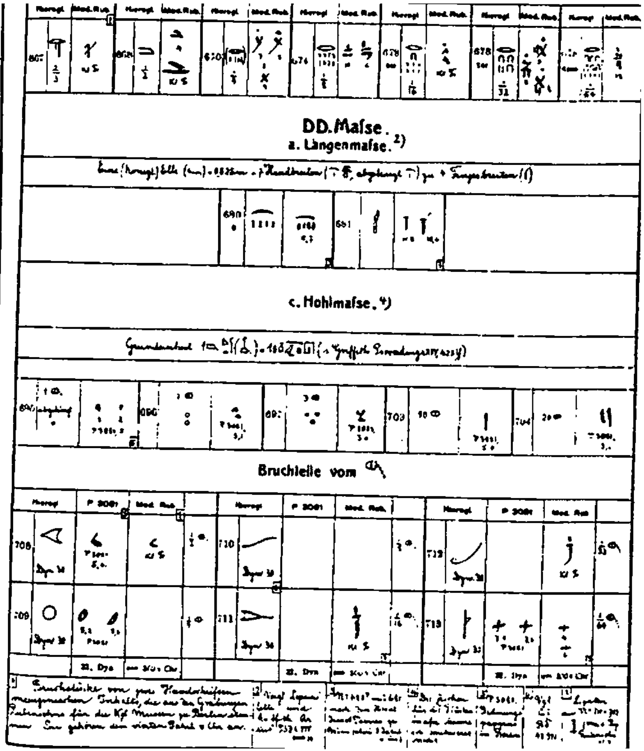
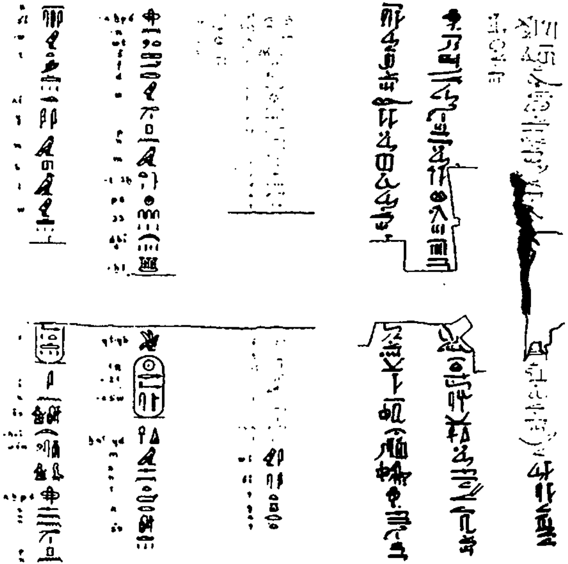


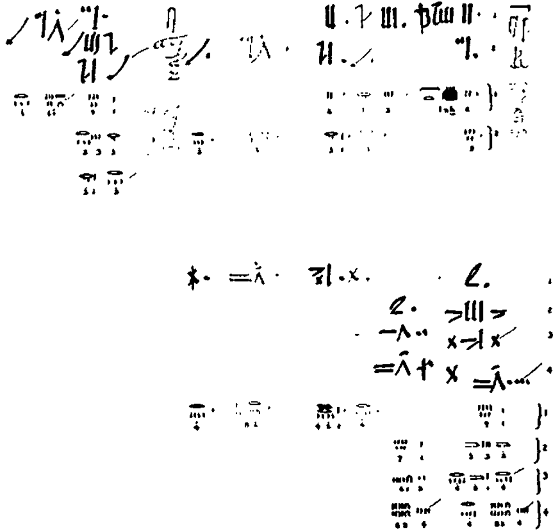
Fig. IV.1j 100s and 1000s; fractions; length and dry measures; Horus-eye fractions of a heqat, *ibid.*, pp. 61, 63.

TITLE-PAGE



Photograph 1, Columns 1–3   B. M. Facsimile, Plate 1

2 DIVIDED BY 3, 5, AND 7



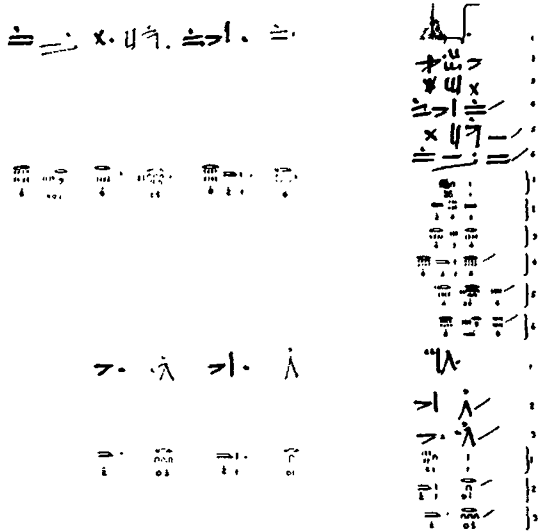
Photograph 1, Registers 1–2   B. M. Facsimile, Plate 1

Plate 3

2 DIVIDED BY 9 AND 11

Photograph 1, Registers 3-4   B. M. Facsimile, Plate 1

2 DIVIDED BY 13 AND 15



Photograph 1, Registers 5-6   B. M. Facsimile, Plate 1

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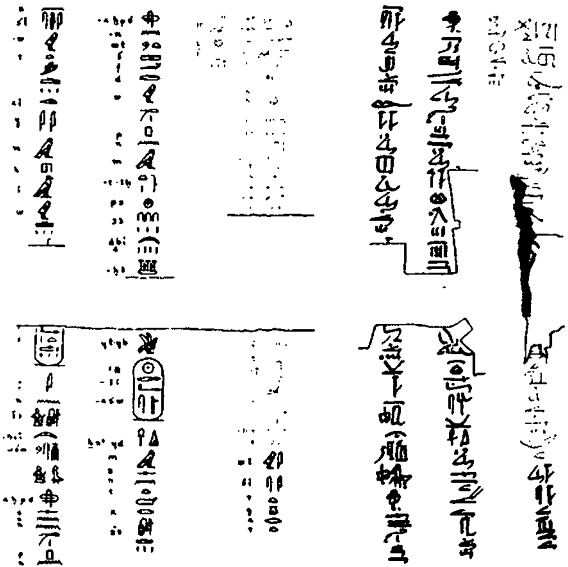
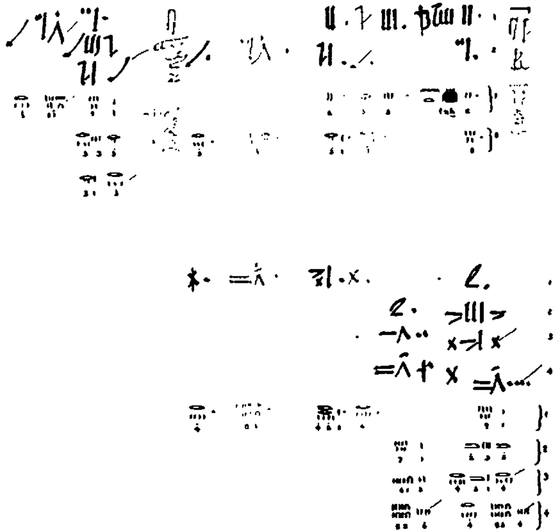


Photograph 1, Columns 1-3 B. M. Facsimile, Plate 1

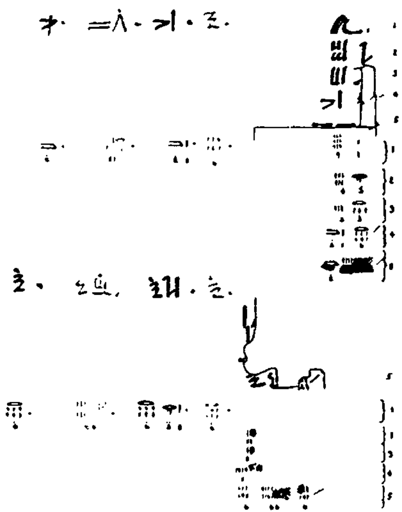
2 DIVIDED BY 3, 5, AND 7



Photograph 1, Registers 1-2 B. M. Facsimile, Plate 1

Plate 3

2 DIVIDED BY 9 AND 11



Photograph 1, Registers 3-4   B. M. Facsimile, Plate 1

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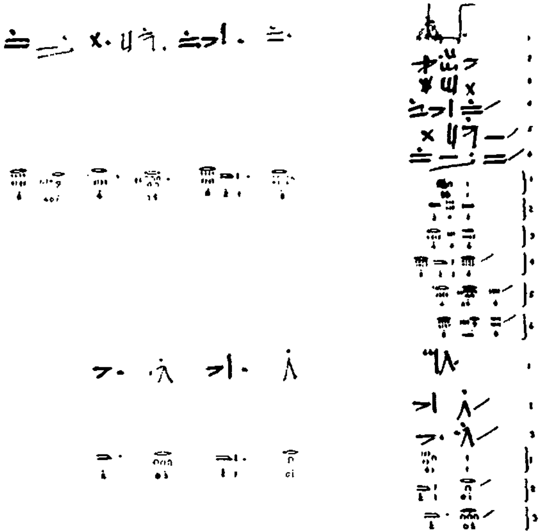
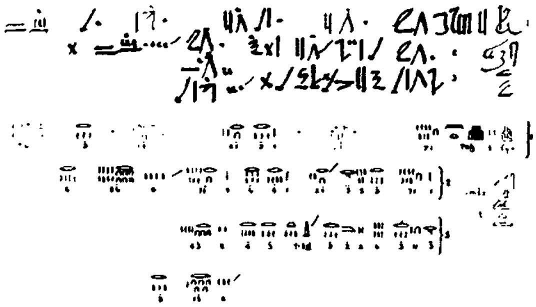
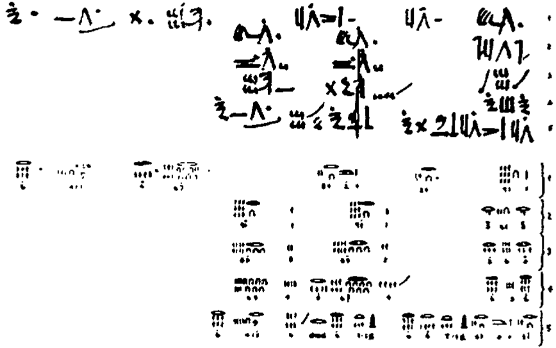


Photograph 1, Registers 5-6   B. M. Facsimile, Plate 1

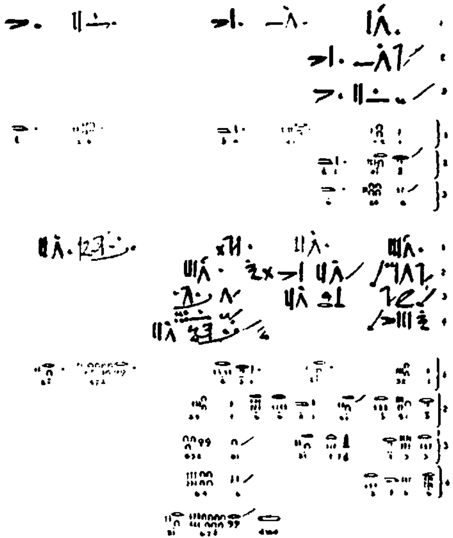


Photograph 11, Register 1   B. M. Facsimile, Plate 11



Photograph 11, Register 2   B. M. Facsimile, Plate 11

Fig. IV.2c Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 5-6.



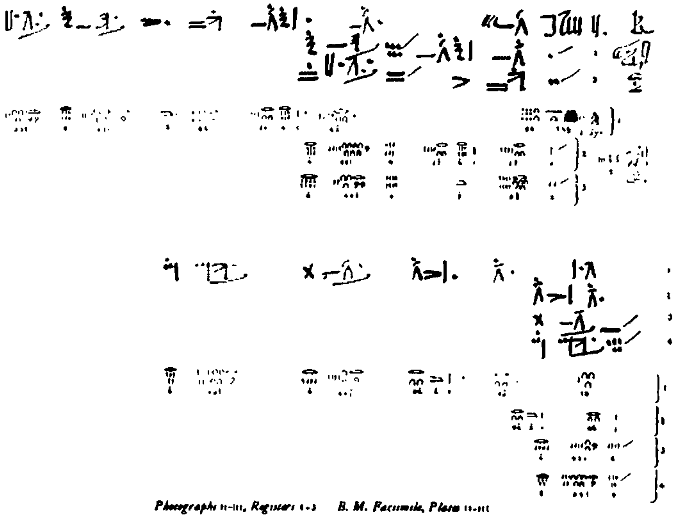
Photograph 11, Registers 3-4 *B. M. Facsimile, Plate 11*

Photograph 11, Registers 5-6 *B. M. Facsimile, Plate 11*

Fig. IV.2d Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 7-8.

Plate 5

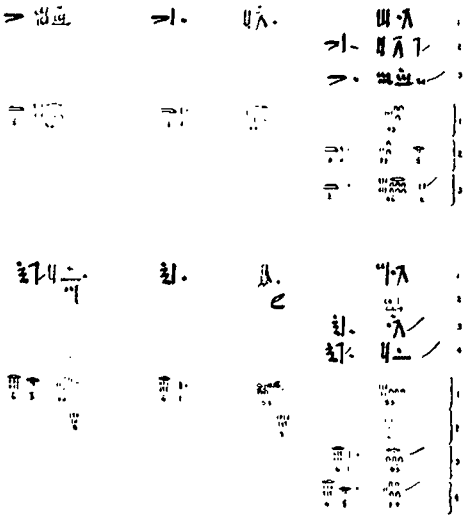
2 DIVIDED BY 29 AND 31



Photographs II–III, *Registers* 1–3   *B. M. Facsimile, Plates* II–III

Plate 10

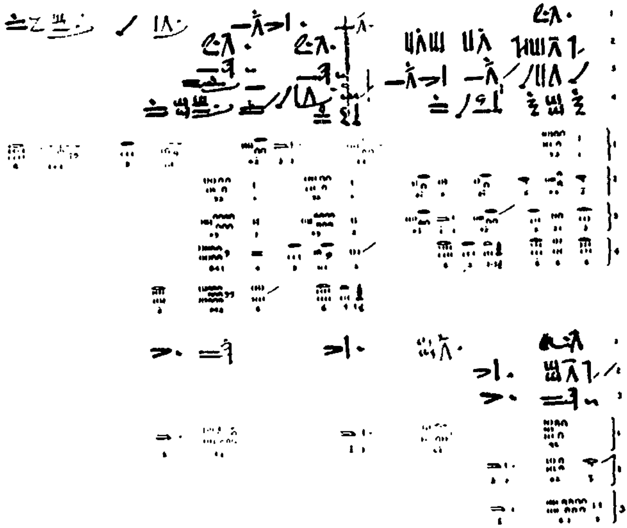
2 DIVIDED BY 33 AND 35



Photograph III, *Registers* 3–4   *B. M. Facsimile, Plates* II–III

Fig. IV.2e Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 9-10.

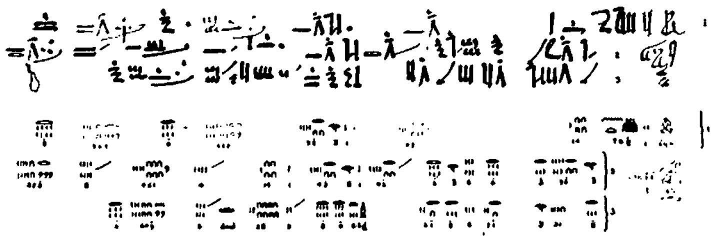
2 DIVIDED BY 37 AND 39



Photograph III, Registers 5-6   *B. M. Facsimile, Plates II-III*

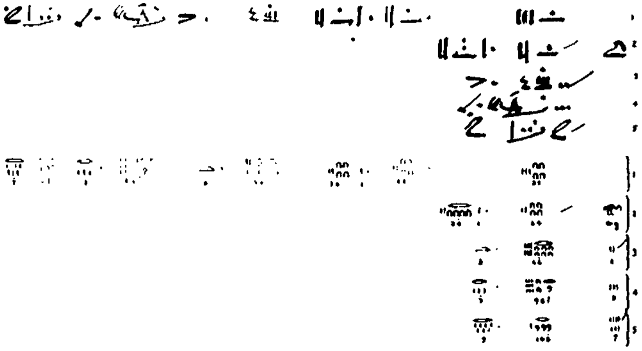
Plate 1

2 DIVIDED BY 41



Photographs III-IV, Register 1   *B. M. Facsimile, Plates III-IV*

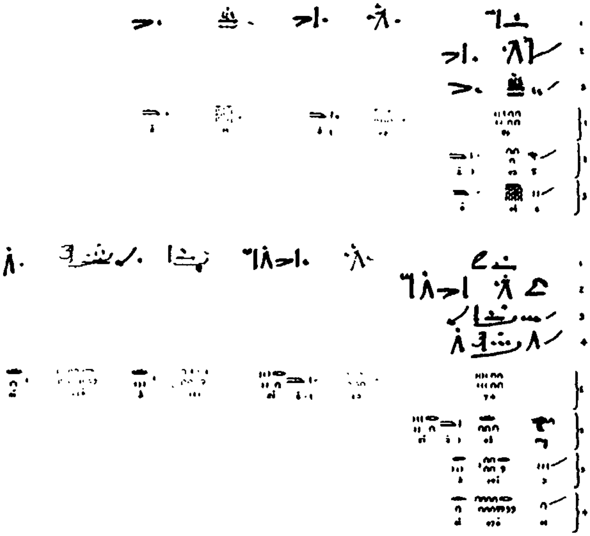
Fig. IV.2f Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 11-12.



Photograph iv, Register 2 B. M. Facsimile, Plates iii-iv

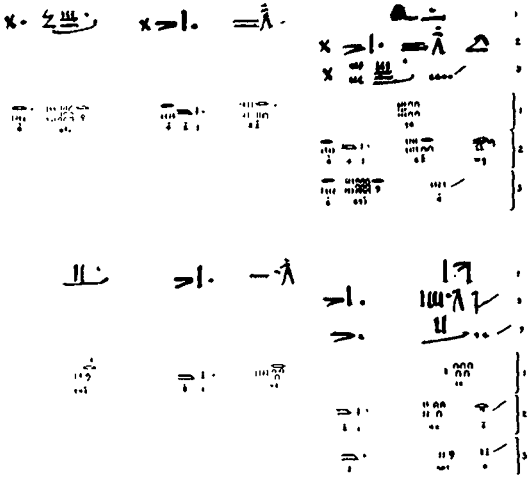
Plate 14

2 DIVIDED BY 45 AND 47



Photograph iv, Registers 3-4 B. M. Facsimile, Plates iii-iv

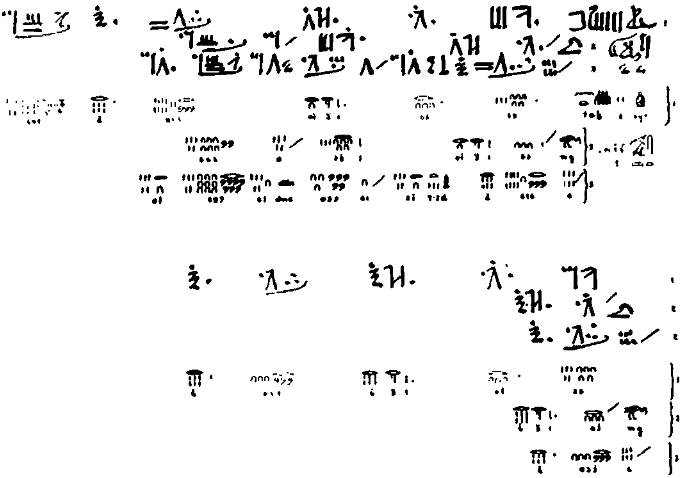
Fig. IV.2g Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 13-14.



Photograph IV, Registers 5-6   B. M. Facsimile, Plates III-IV

Plate

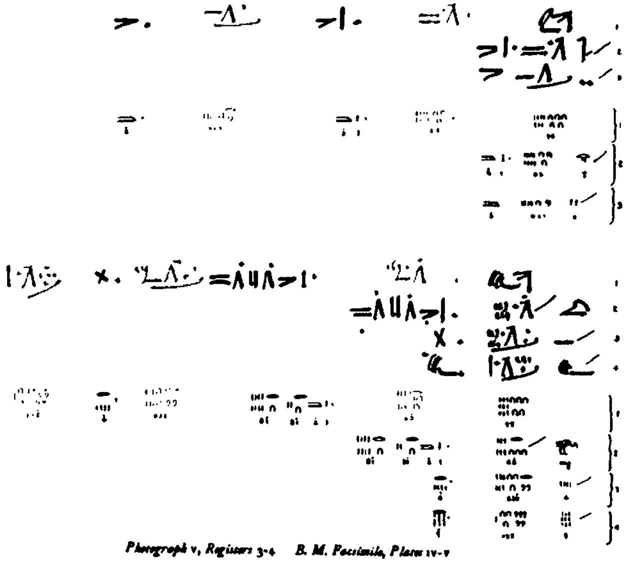
2 DIVIDED BY 53 AND 55



Photographs IV-V, Registers 1-2   B. M. Facsimile, Plates IV-V

Fig. IV.2h Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 15-16.

2 DIVIDED BY 57 AND 59



Photograph v, Registers 3-4   B. M. Facsimile, Plates iv-v

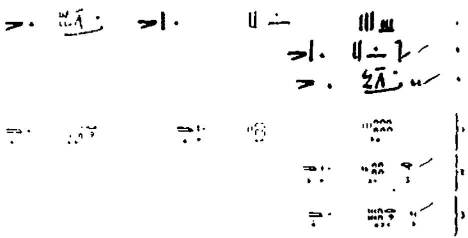
2 DIVIDED BY 61

Photographs iv-v, Register 5   B. M. Facsimile, Plates iv-v

Fig. IV.21 Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 17-18.

Plate 19

2 DIVIDED BY 63



Photograph v, Register 6   B. M. Facsimile, Plates iv–v

Plate 20

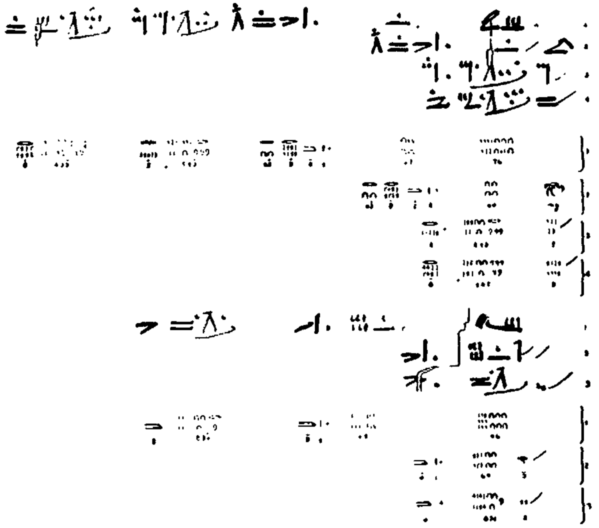
2 DIVIDED BY 65

Photographs v–vi, Register 1   B. M. Facsimile, Plate v

Fig. IV.2j Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 19-20.

Plate

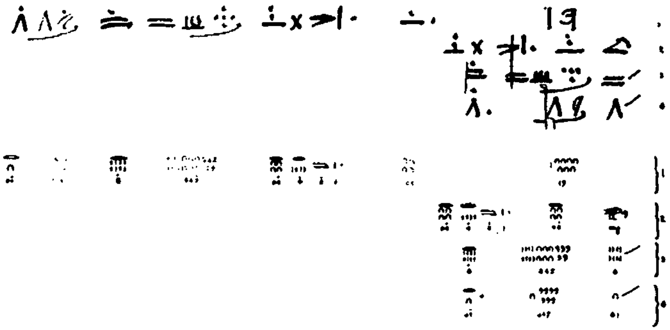
2 DIVIDED BY 67 AND 69



Photograph VI, *Registers* 2-3   *B. M. Facsimile, Plates* V-VI

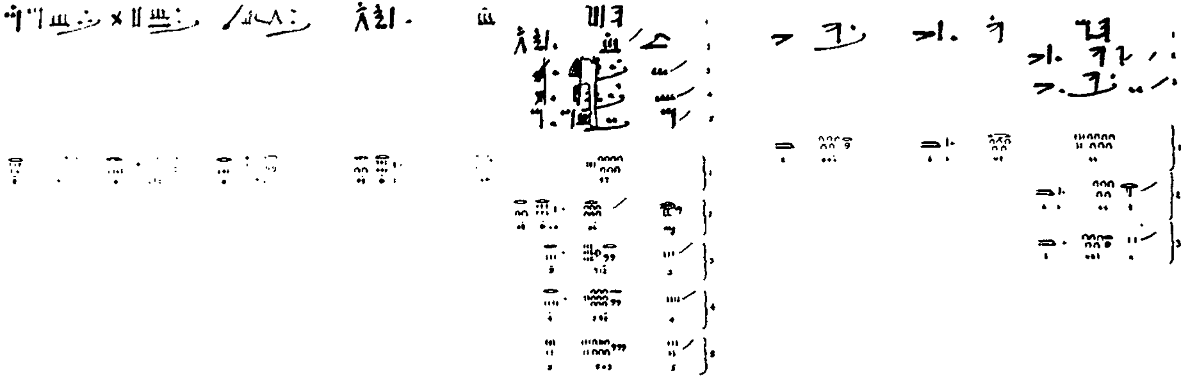
Plate

2 DIVIDED BY 71



Photograph VI, *Register* 4   *B. M. Facsimile, Plates* V-VI

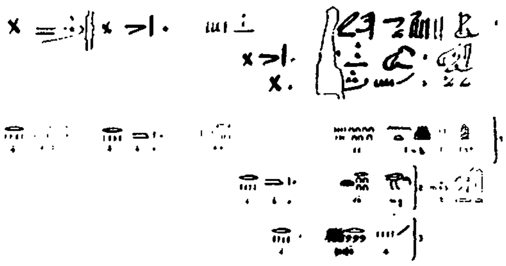
Fig. IV.2k Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 21-22.



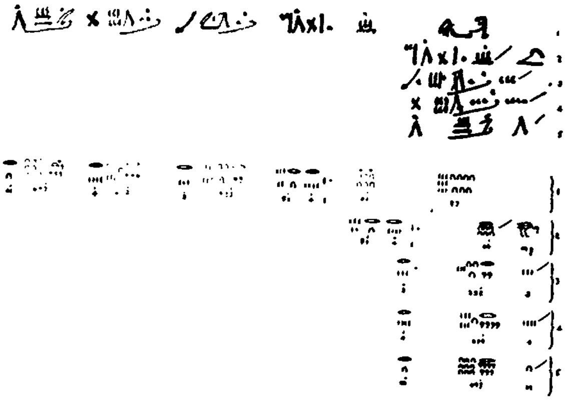
Photograph vi, Register 5   B. M. Facsimile, Plates v-vi

Photograph vi, Register 6   B. M. Facsimile, Plate v

Fig. IV.2L Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 23-24.

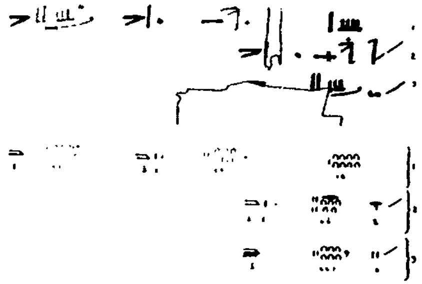


Photographs VI–VII, *Register* 1 *B. M. Facsimile, Plate* VI

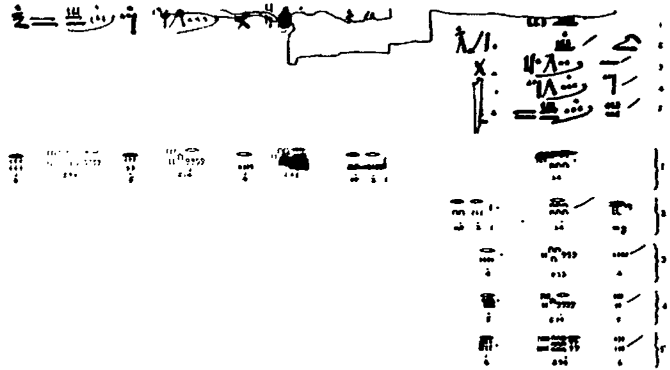


Photographs VI–VII, *Register* 2 *B. M. Facsimile, Plate* VI

Fig. IV.2m Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*. Vol. 2 (Oberlin, Ohio, 1929), Plates 25-26.



Photographs VI–VII, Register 3   B. M. Facsimile, Plate VI



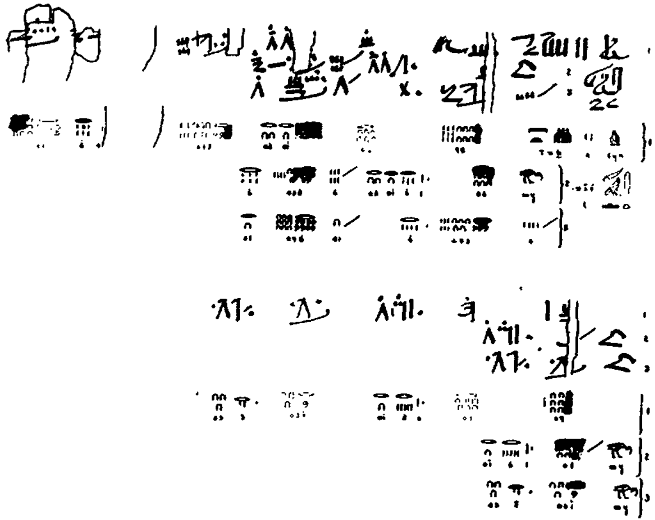
Photographs VI–VII, Register 4   B. M. Facsimile, Plate VI

Fig. IV.2n Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 27-28.

2 DIVIDED BY 85 AND 87

Photographs VI–VII, *Registers* 5–6   *B. M. Facsimile, Plate* VI

2 DIVIDED BY 89 AND 91



Photograph VIII, *Registers* 1–2   *B. M. Facsimile, Plates* VI–VII
Also Photograph IX, *N. Y. Historical Society Fragments*

Fig. IV.2o Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind*

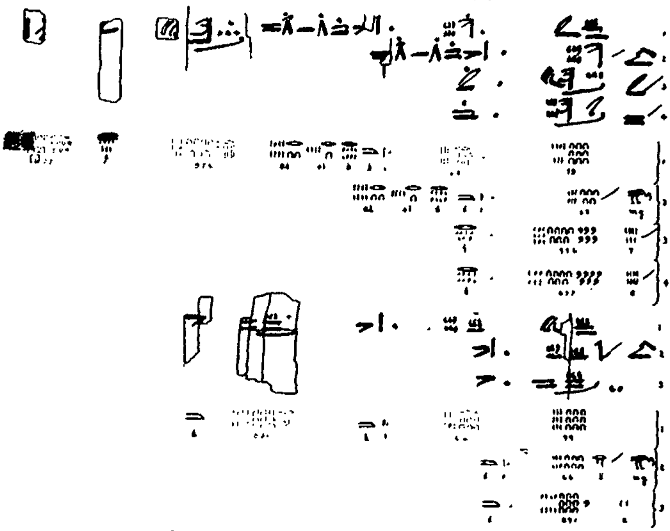
Plat

2 DIVIDED BY 93 AND 95

Photograph VIII, *Registers* 3-4 *B. M. Facsimile, Plates* VI-VII *Also Photograph* IX, *N. Y. Historical Society Fragments*

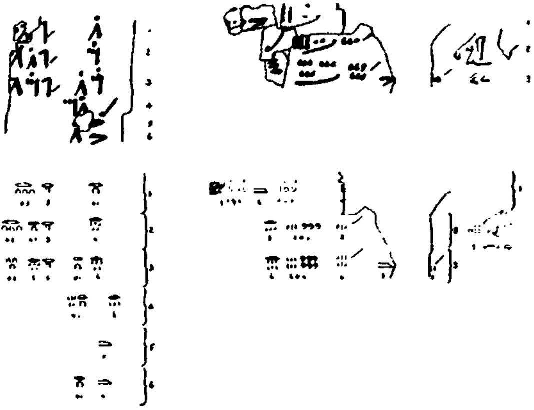
Plate

2 DIVIDED BY 97 AND 99



Photograph VIII, *Registers* 5-6 *B. M. Facsimile, Plates* VI-VIII
Also Photograph IX, *N. Y. Historical Society Fragments*

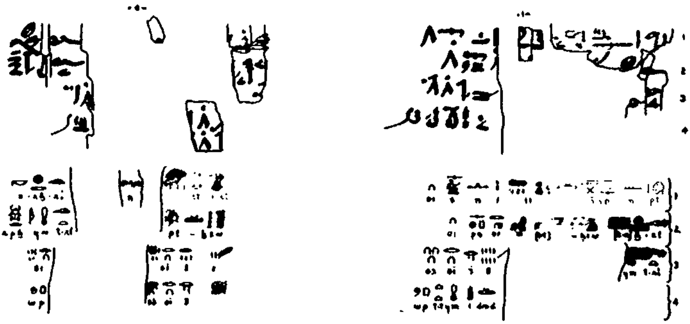
Fig. IV.2p Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 31-32.



Photograph ix, Registers 1-2   N. Y. Historical Society Fragments

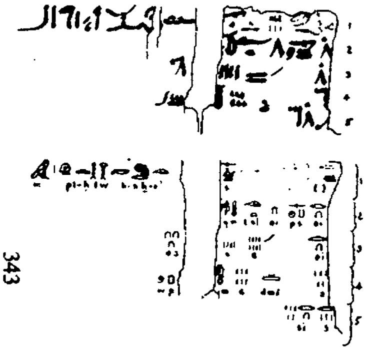
Plate

PROBLEMS 1 AND 2



Photograph i, Registers 1-3   B. M. Facsimile, Plate vii
Also Photograph ix, N. Y. Historical Society Fragments

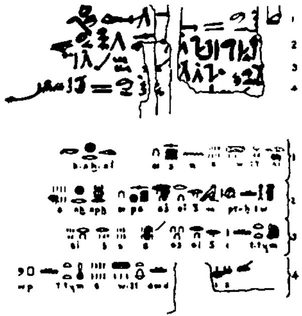
Fig. IV.2q Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 33-34.



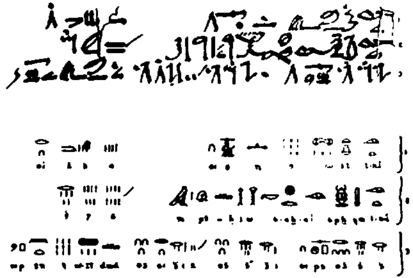
Photograph x, Register 4 B. M. Facsimile, Plate vii
Also Photograph ix, N.Y. Historical Society Fragments

Photograph x, Register 5 B. M. Facsimile, Plate vii
Also Photograph ix, N.Y. Historical Society Fragments

Fig. IV.2r Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 35-36.

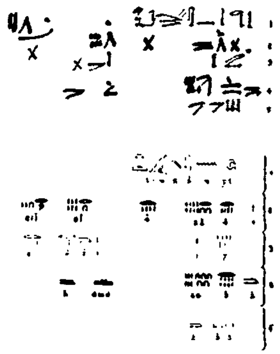


Photograph x, Register 6 B. M. Facsimile, Plate VII
Also Photograph IX, N. Y. Historical Society Fragments

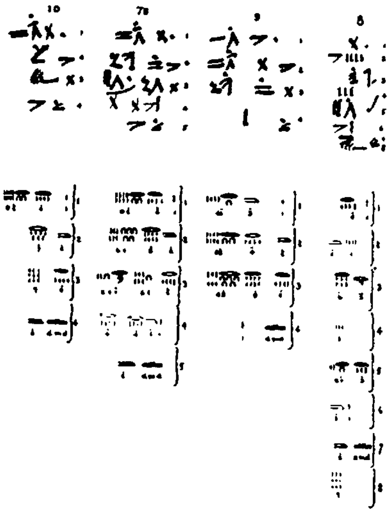


Photograph x, Register 1 B. M. Facsimile, Plates VII–VIII

Fig. IV.2s Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 37-38.

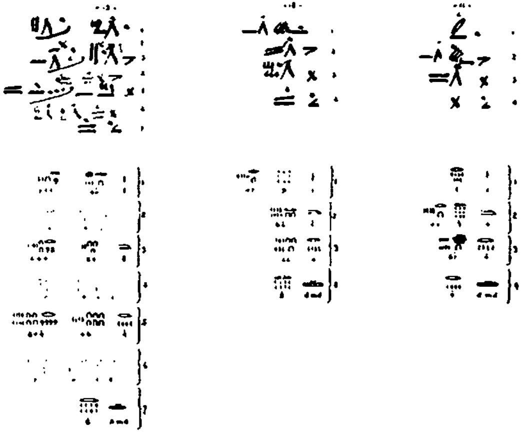


Photograph 1, Register 2   *B. M. Facsimile, Plates* VII–VIII

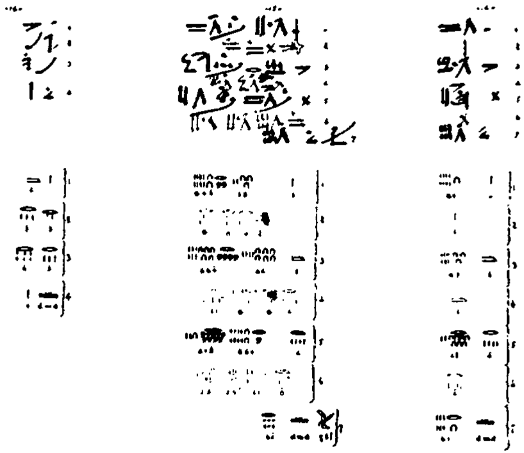


Photograph 1, Register 3   *B. M. Facsimile, Plates* VII–VIII

Fig. IV.2t Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 39-40.

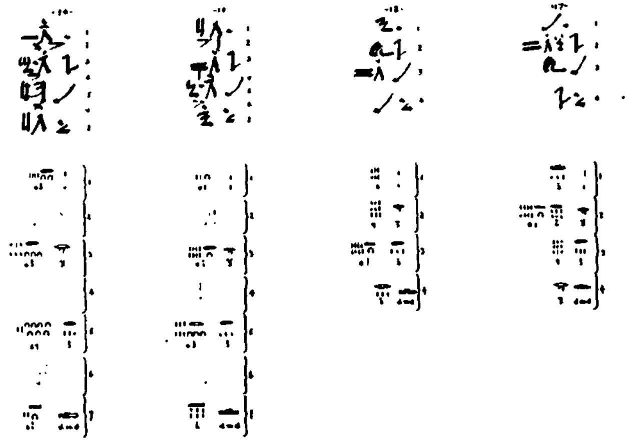


Photograph 1, Register 4   *B. M. Facsimile, Plates VII–VIII*



Photograph 1, Registers 4–5   *B. M. Facsimile, Plates VII–VIII*

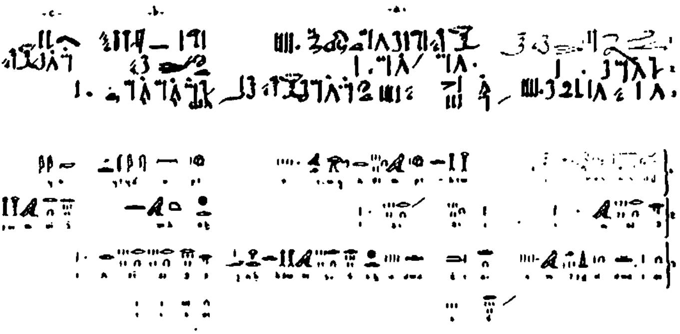
Fig. IV.2u Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 41-42.



Photograph x, Registers 5-6 B. M. Facsimile, Plates vii-viii

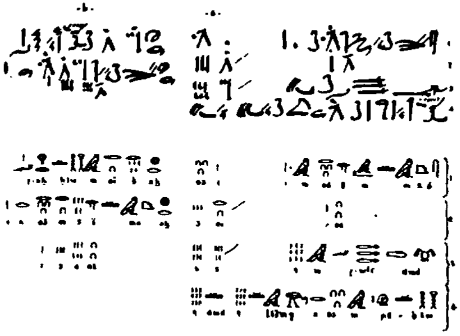
Plate

PROBLEM 21

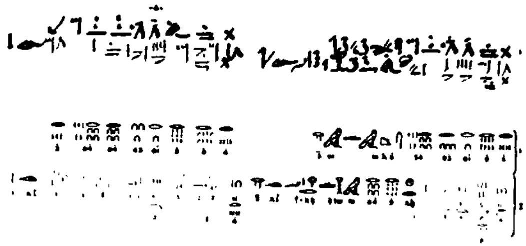


Photograph xi, Register 1 B. M. Facsimile, Plate viii

Fig. IV.2v Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 43-44.

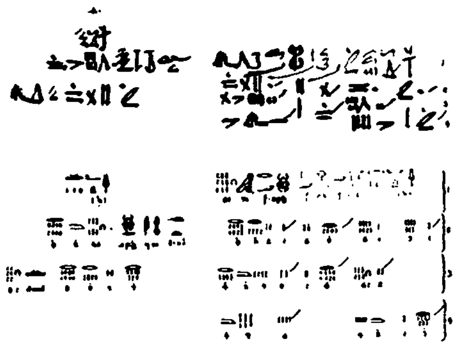


Photograph xı, Register 2   B. M. Facsimile, Plate vııı

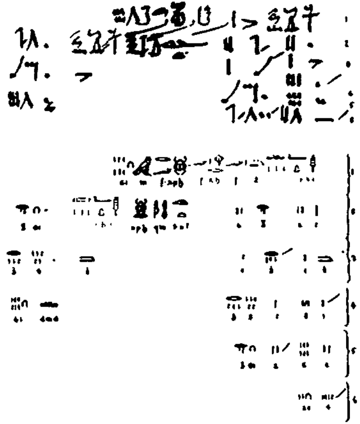


Photograph xı, Register 2   B. M. Facsimile, Plate vııı

Fig. IV.2w Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 45-46.

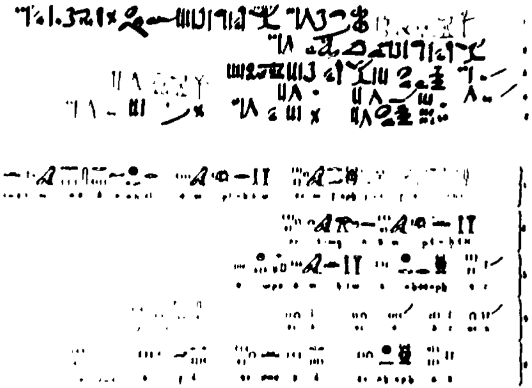


Photograph XII, *Register* 1 *B. M. Facsimile, Plate* IX

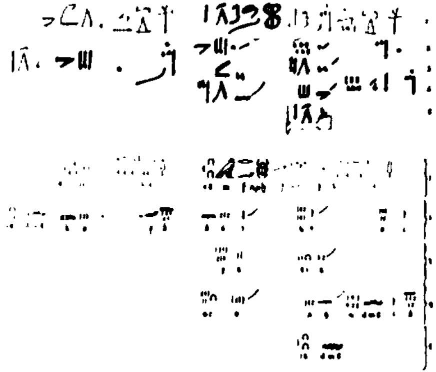


Photograph XII, *Register* 2 *B. M. Facsimile, Plate* IX

Fig. IV.2x Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 47-48.

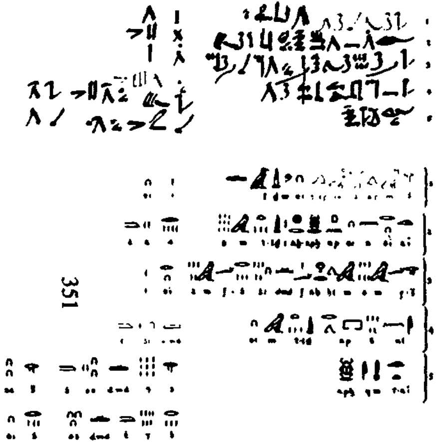


Photograph xii, Register 3 B. M. Facsimile, Plate 12

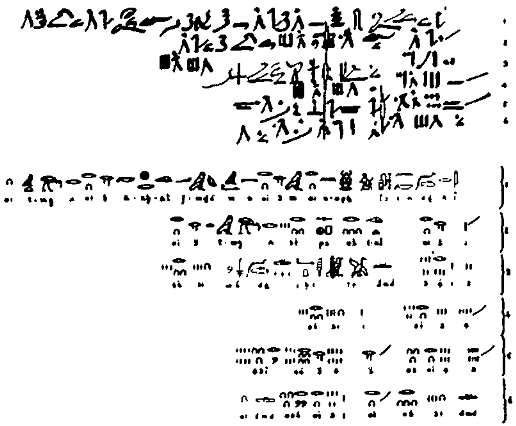


Photograph xii, Register 4 B. M. Facsimile, Plate 12

Fig. IV.2y Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 49-50.

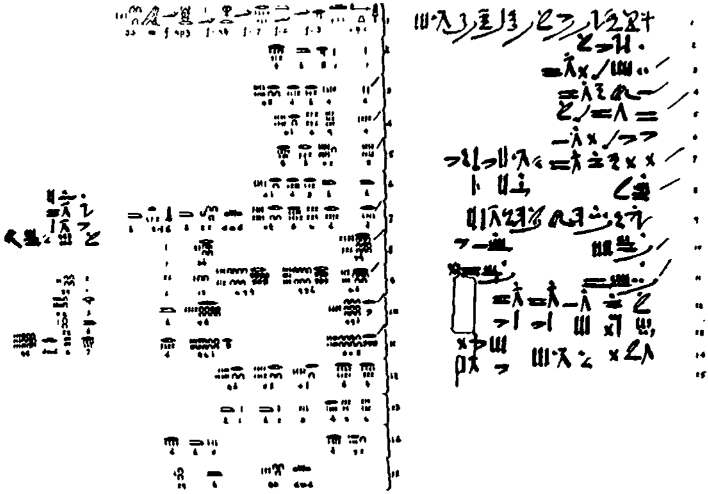


Photograph XII, Register 5   B. M. Facsimile, Plate 12



Photograph XII, Register 6   B. M. Facsimile, Plate 12

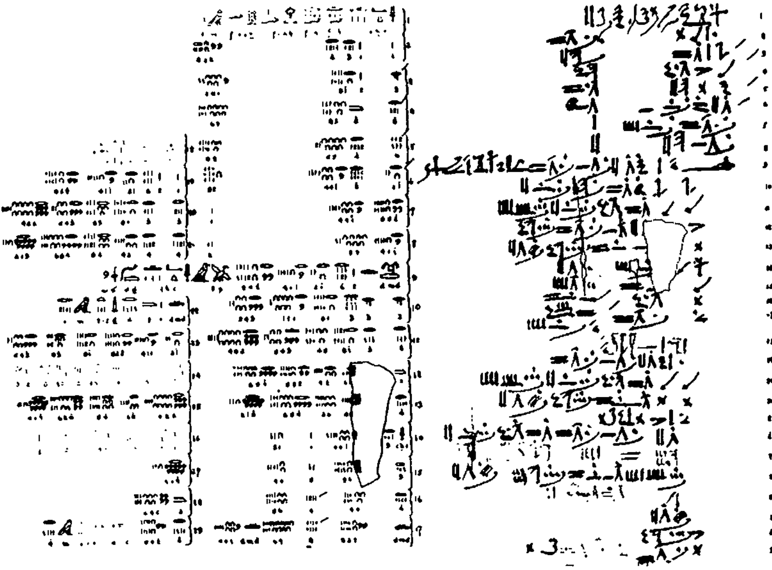
Fig. IV.2z Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 51-52.



Photograph xiii, Registers 1-4 *B. M. Facsimile, Plate ix*
Also Photograph xv, Register 4 *B. M. Facsimile, Plate xi*

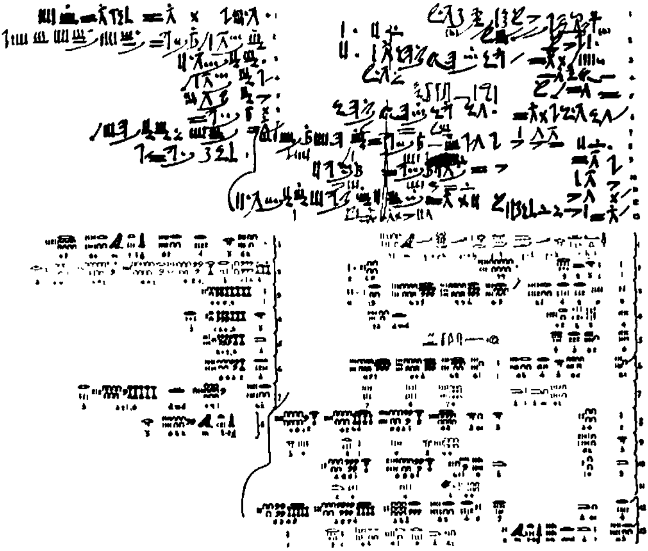
Plate 54

PROBLEM 32



Photograph xiii, Registers 1-6 *B. M. Facsimile, Plates ix-x*

Fig. IV.2aa Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 53-54.



Photograph xiv, *Registers* 5-6 *B. M. Facsimile, Plates* x-xi

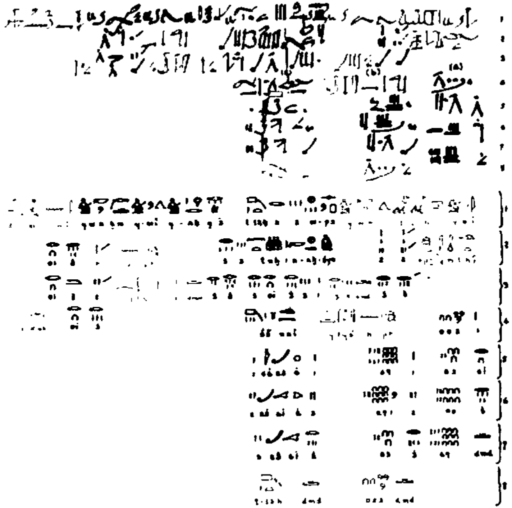
Plate

PROBLEM 34

Photograph xv, *Registers* 5-6 *B. M. Facsimile, Plate* xi

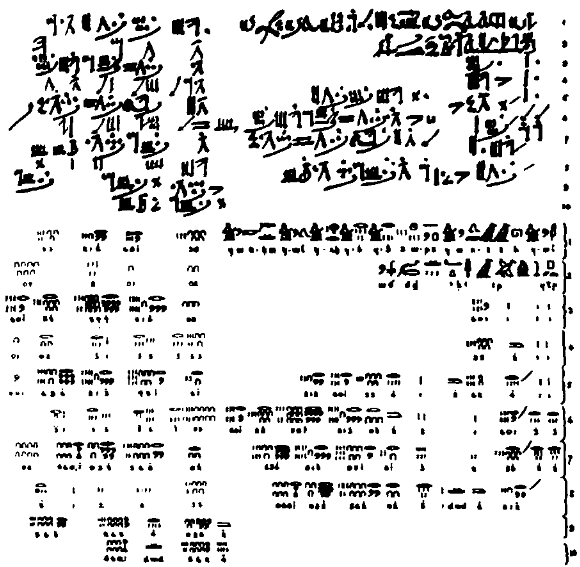
Fig. IV.2bb Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 55-56.

PROBLEM 35



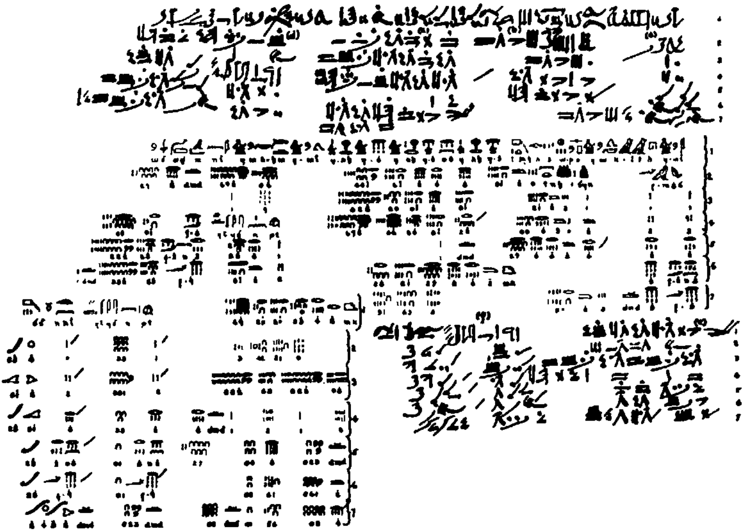
Photograph xiv, Registers 1-2   B. M. Facsimile, Plate 1

PROBLEM 36



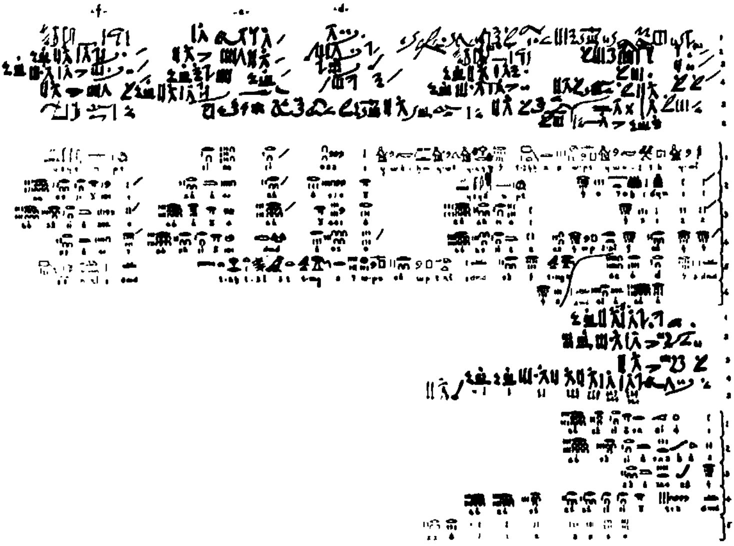
Photograph xv, Registers 1-2   B. M. Facsimile, Plates 1-11

PROBLEM 37



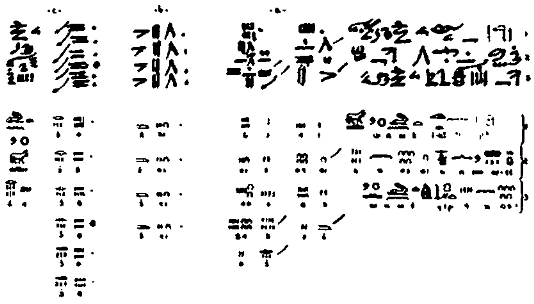
Photographs xiv-xv, *Register* 3   *B. M. Facsimile, Plates* x-xi

PROBLEM 38

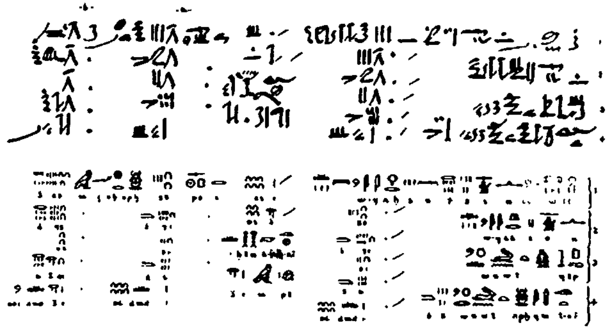


Photographs xiv-xv, *Register* 4   *B. M. Facsimile, Plates* x-xi

Fig. IV.2dd Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 59-60.



Photographs XVI–XVII, *Register* 1 *B. M. Facsimile, Plates* XI–XII

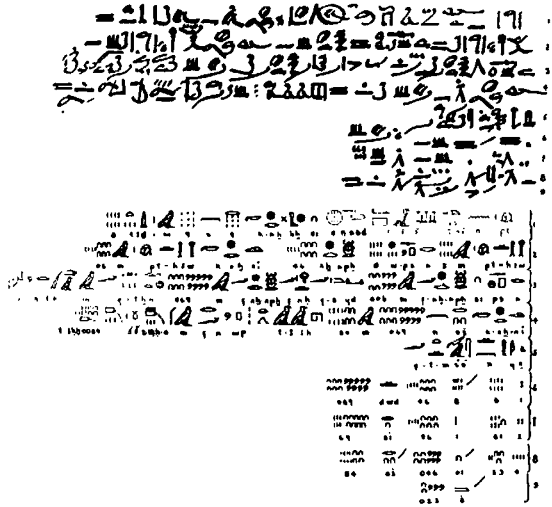


Photographs XVI–XVII, *Register* 2 *B. M. Facsimile, Plates* XI–XII

Fig. IV.2ee Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 61-62.

Plate 63

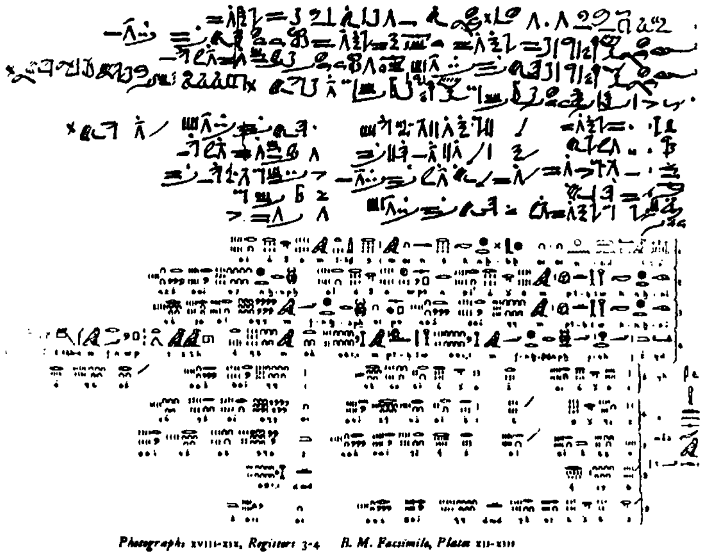
PROBLEM 41



Photograph XVIII, *Registers* 1-2 *B. M. Facsimile, Plates* XII-XIII

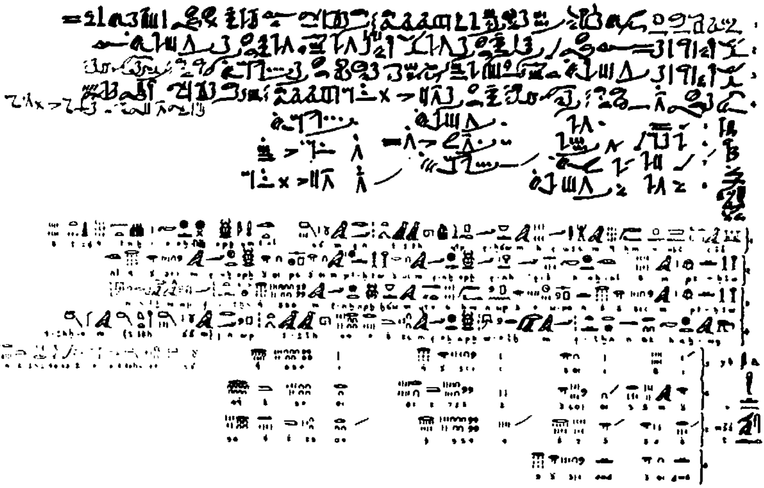
Plate 64

PROBLEM 42



Photographs XVIII-XIX, *Registers* 3-4 *B. M. Facsimile, Plates* XII-XIII

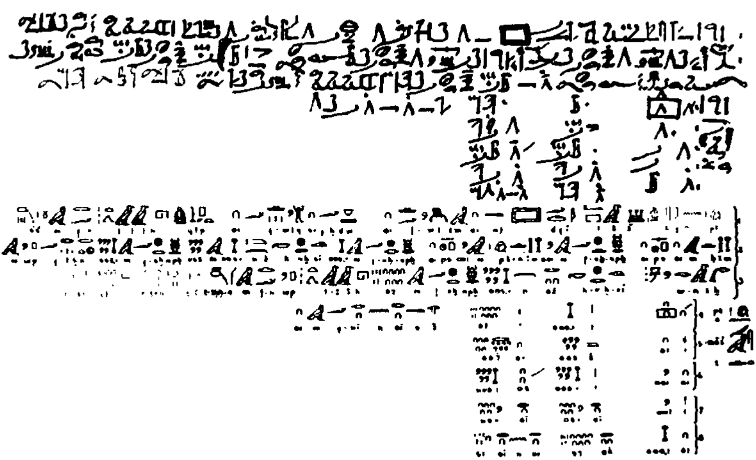
PROBLEM 43

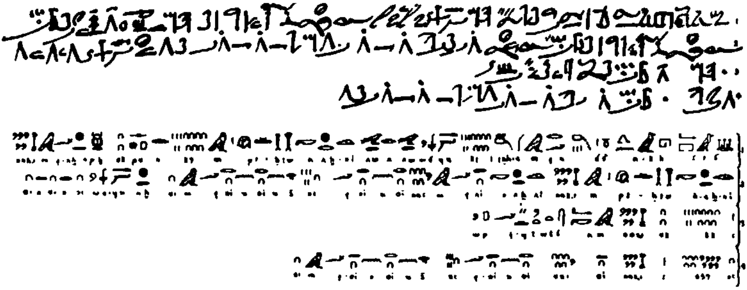


Photographs xviii–xix, *Registers* 5–6   *B. M. Facsimile, Plates* xii–xiii

Plate 66

PROBLEM 44

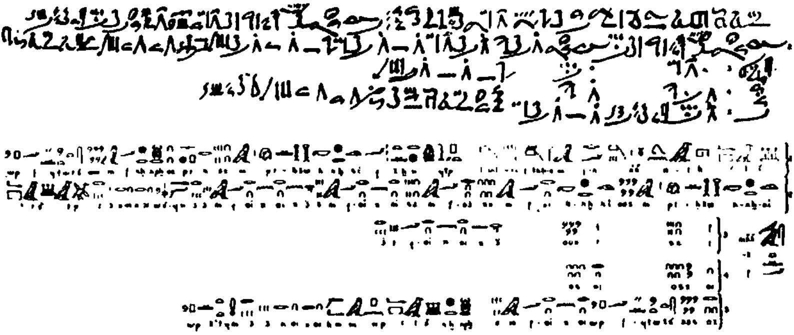




Photographs xix-xx, *Register* 3   *B. M. Facsimile, Plates* xiii-xiv

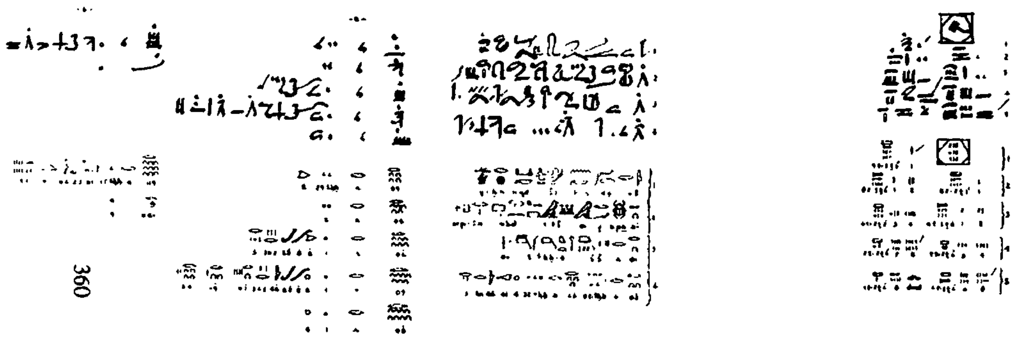
Plate 68

PROBLEM 46



Photographs xix-xx, *Register* 4   *B. M. Facsimile, Plates* xiii-xiv

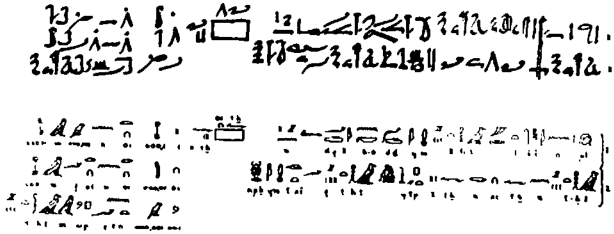
Fig. IV.2hh Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 67-68.



Photographs XIX–XX, *Register* 5 *B. M. Facsimile, Plates* XIII–XIV

Photograph XIX, *Register* 6 *B. M. Facsimile, Plates* XIII–XIV

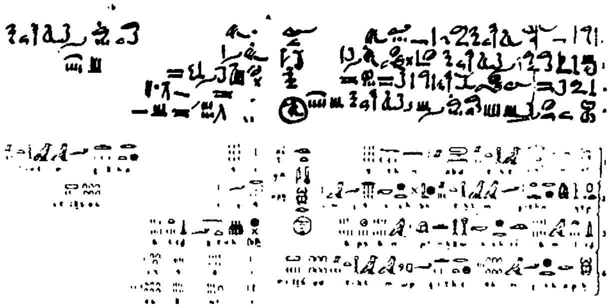
Fig. IV.2ii Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 69-70.



Photographs xx-xxi, Register 1 B. M. Facsimile, Plate xiv

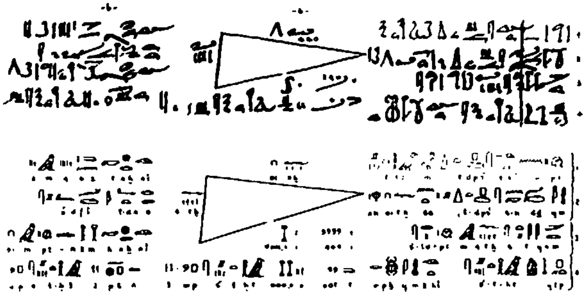
Plate 72

PROBLEM 50



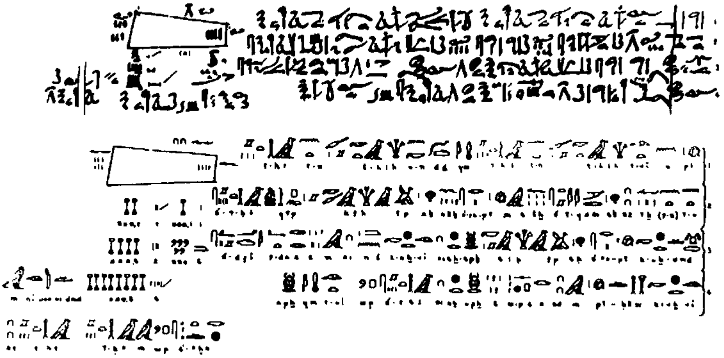
Photographs xx-xxi, Register 2 B. M. Facsimile, Plate xiv

Fig. IV.2jj Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 71-72.



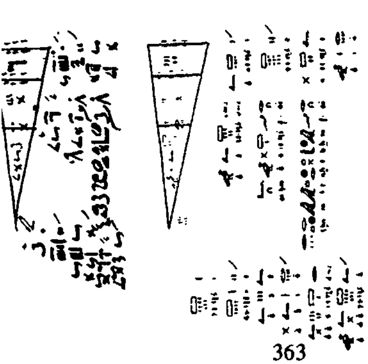
Photographs xx-xxi, *Register* 3   *B. M. Facsimile, Plate* xiv

PROBLEM 52

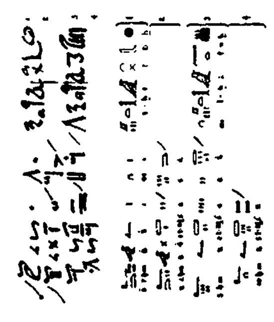


Photographs xx-xxi, *Register* 4   *B. M. Facsimile, Plates* xiv-xv

Fig. IV.2kk Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 73-74.

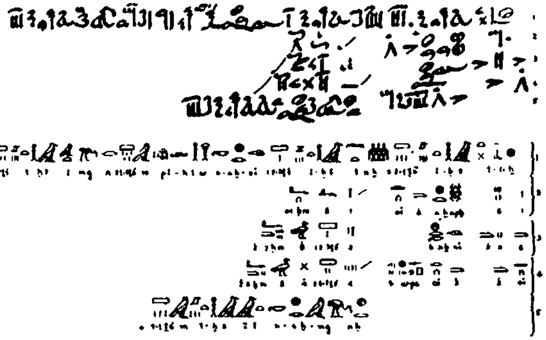


Photographs xx-xxi, Register 5   B. M. Facsimile, Plate xiv



Photograph xx, Register 5   B. M. Facsimile, Plate xiv

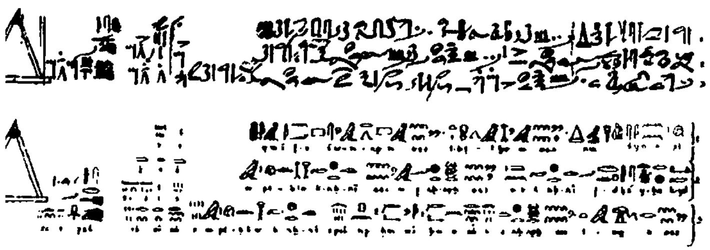
Fig. IV.2LL Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 75-76.



Photographs xx-xxi, Register 6   B. M. Facsimile, Plate xiv

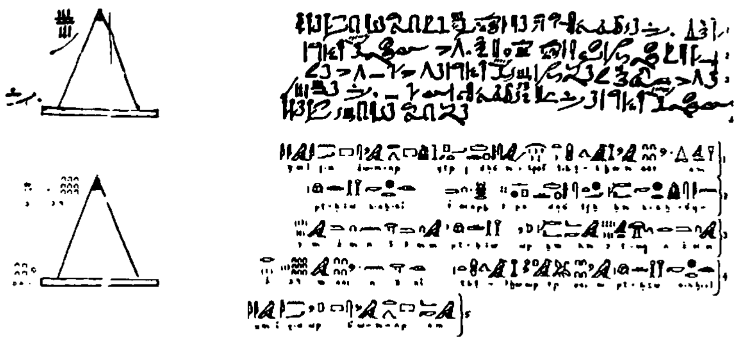
Plate 78

PROBLEM 56



Photographs xxi-xxii, Register 1   B. M. Facsimile, Plate xv

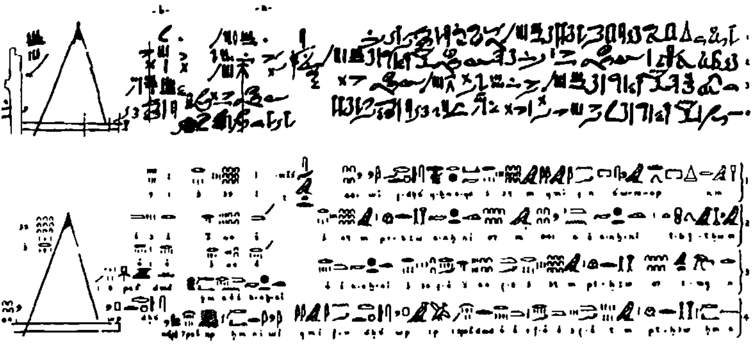
Fig. IV.2mm Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 77-78.



Photographs xxi-xxii, Register 2 *B. M. Facsimile, Plate xv*

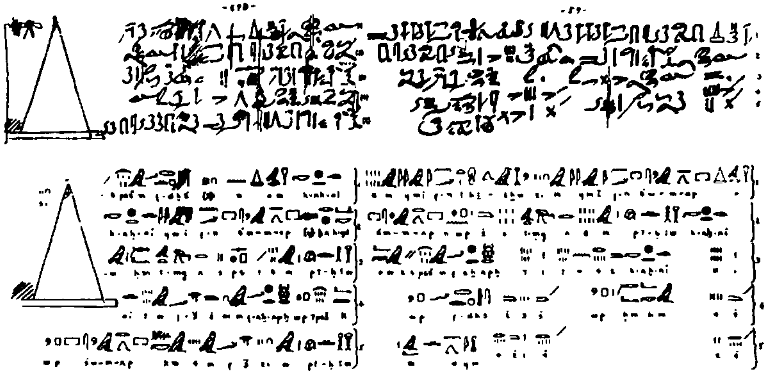
Plate 80

PROBLEM 58



Photographs xxi-xxii, Register 3 *B. M. Facsimile, Plate xv*

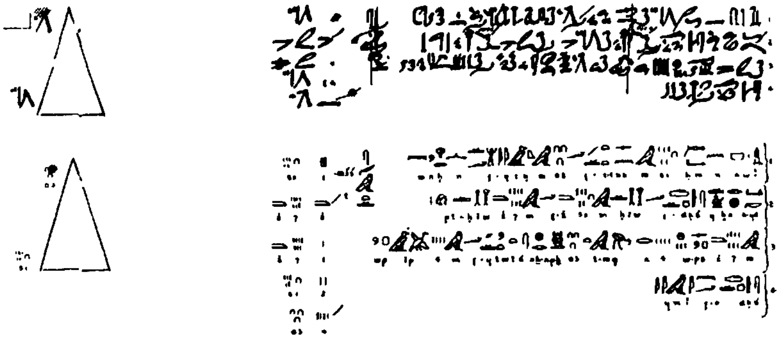
Fig. IV.2nn Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 79-80.



Photographs xxi-xxii, Register 4 B. M. Facsimile, Plate xv

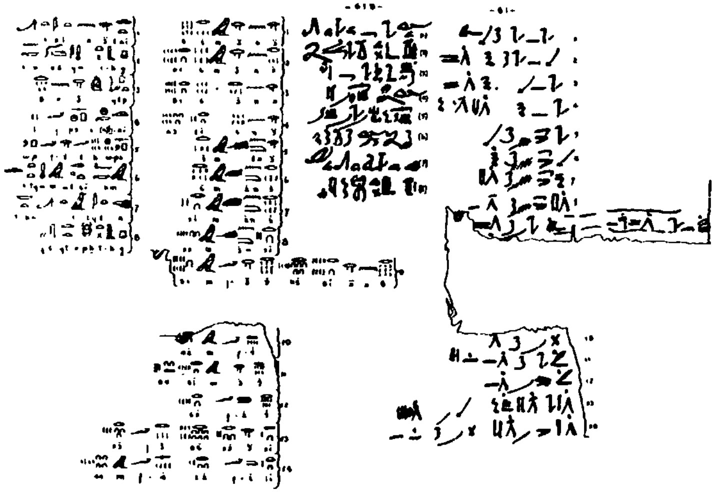
Plate 82

PROBLEM 60



Photographs xxi-xxii, Register 5 B. M. Facsimile, Plate xv

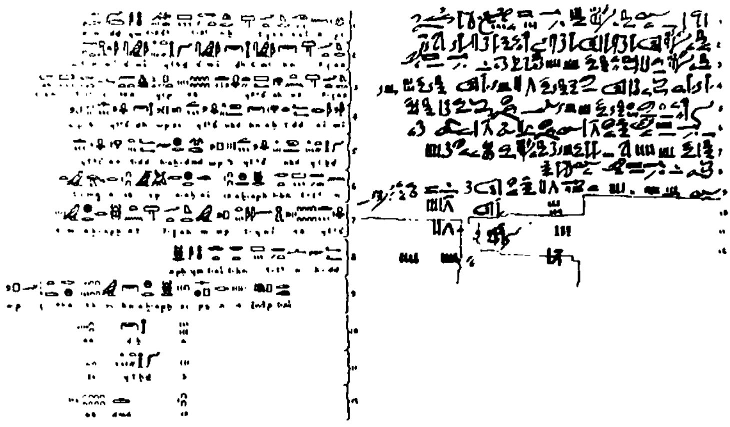
Fig. IV.200 Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 81-82.



Photograph xxiii, Right Half *B. M. Facsimile, Plate xvi*

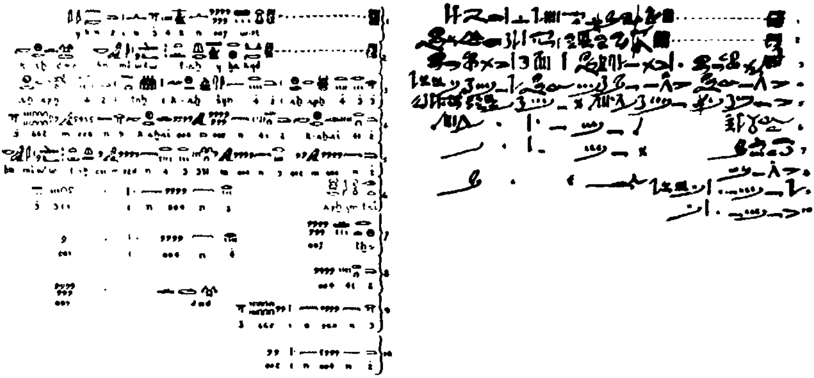
Plate

PROBLEM 62



Photographs xxiii-xxiv, Registers 1-3 *B. M. Facsimile, Plates xvi-xvii*

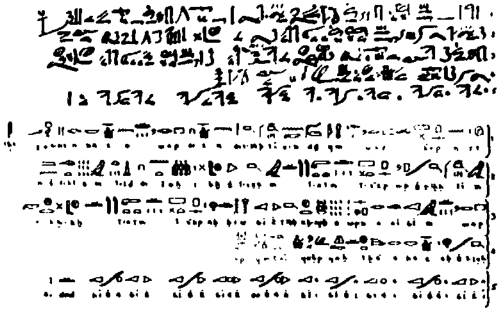
Fig. IV.2pp Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 83-84.



Photographs xxiii-xxiv, *Registers* 4-5 *B.M. Facsimile, Plates* xvi-xvii

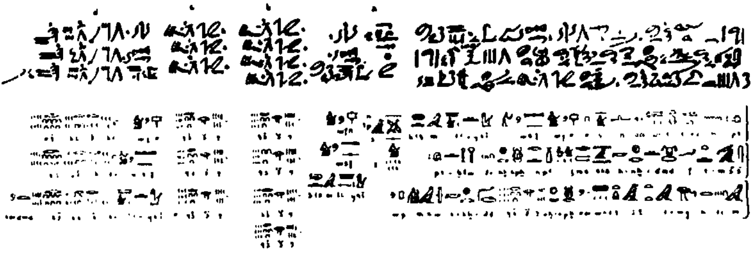
Plate 86

PROBLEM 64



Photographs xxiii-xxiv, *Register* 6 *B.M. Facsimile, Plates* xvi-xvii

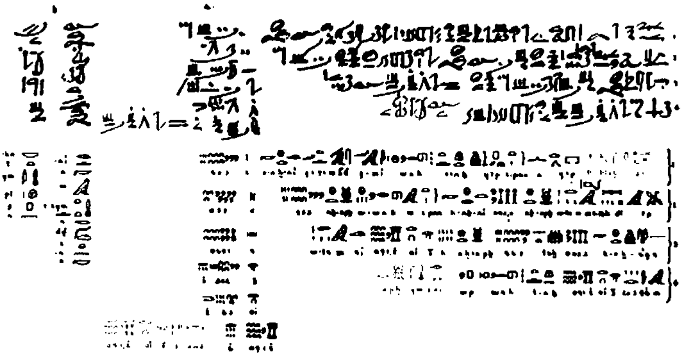
Fig. IV.2qq Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 85-86.



Photographs xxiv-xxv, *Register* 1   *B. M. Facsimile, Plates* xvii-xviii

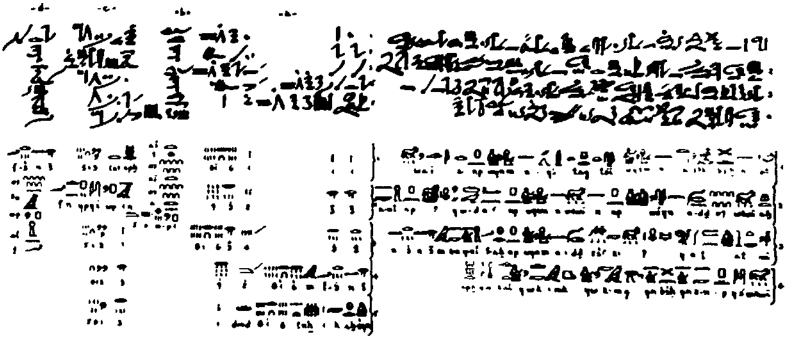
Plate 88

PROBLEM 66



Photographs xxiv-xxv, *Register* 1   *B. M. Facsimile, Plate* xvii

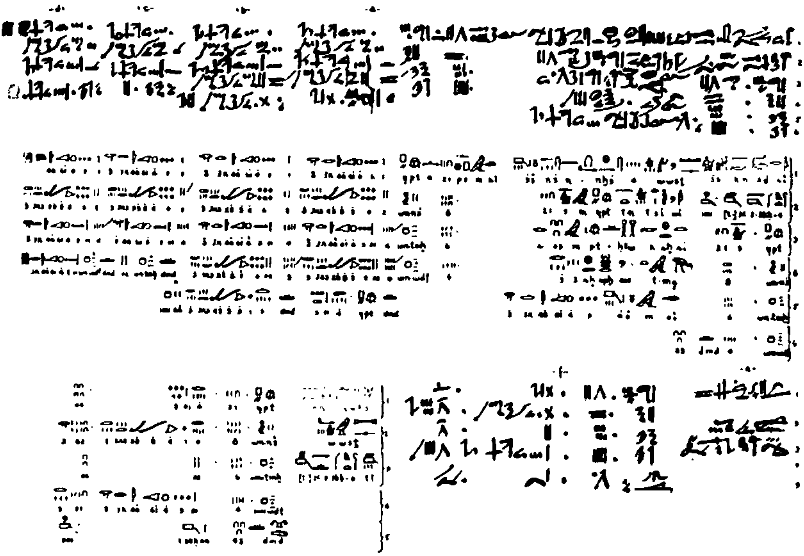
Fig. IV.2rr Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 87-88.



Photographs xxiv-xxv, *Register* 3 *B. M. Facsimile, Plates* xvii-xviii

Plate 90

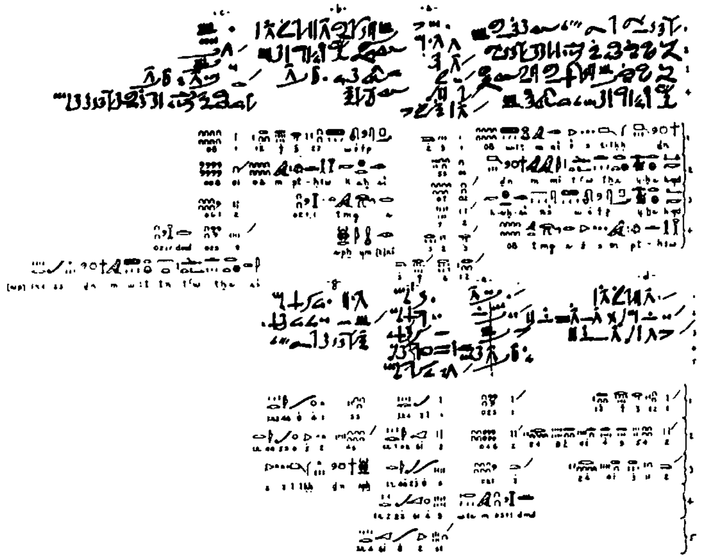
PROBLEM 68



Photographs xxiv-xxvi, *Register* 4 *B. M. Facsimile, Plates* xvii-xviii

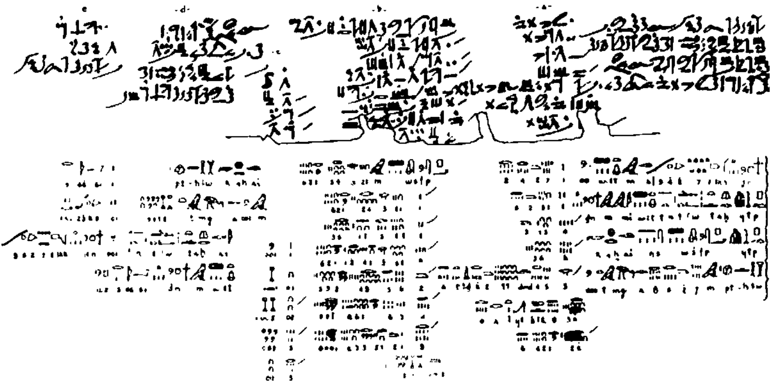
Fig. IV.2ss Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 89-90.

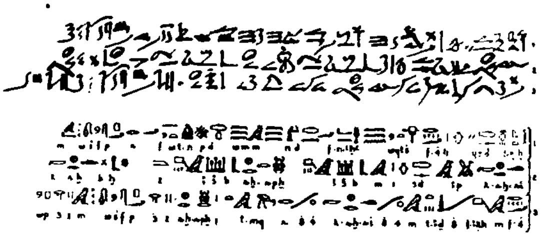
PROBLEM 69



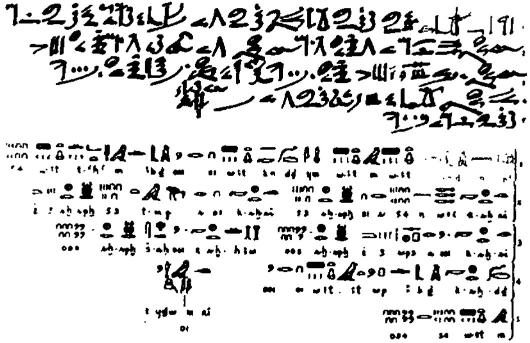
Photographs xxiv-xxv, *Register* 5 *B. M. Facsimile, Plates* xvii-xviii

PROBLEM 70



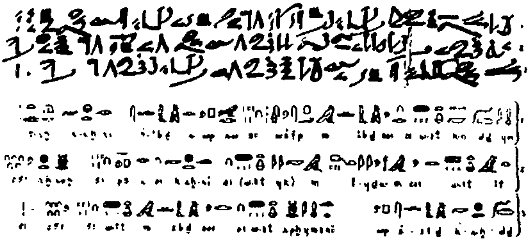


Photographs XXV–XXVI, *Register* 1   *B. M. Facsimile, Plate* XVIII



Photographs XXV–XXVI, *Register* 2   *B. M. Facsimile, Plate* XVIII

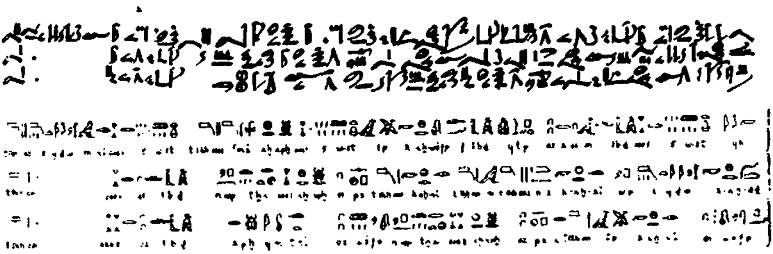
Fig. IV.2uu Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 93-94.



Photographs xxv–xxvi, *Register* 3   *B. M. Facsimile, Plate* xviii

Plate 96

PROBLEM 74



Photographs xxvi–xxvii, *Register* 1   *B. M. Facsimile, Plates* xviii–xix

Fig. IV.2vv Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 95-96.

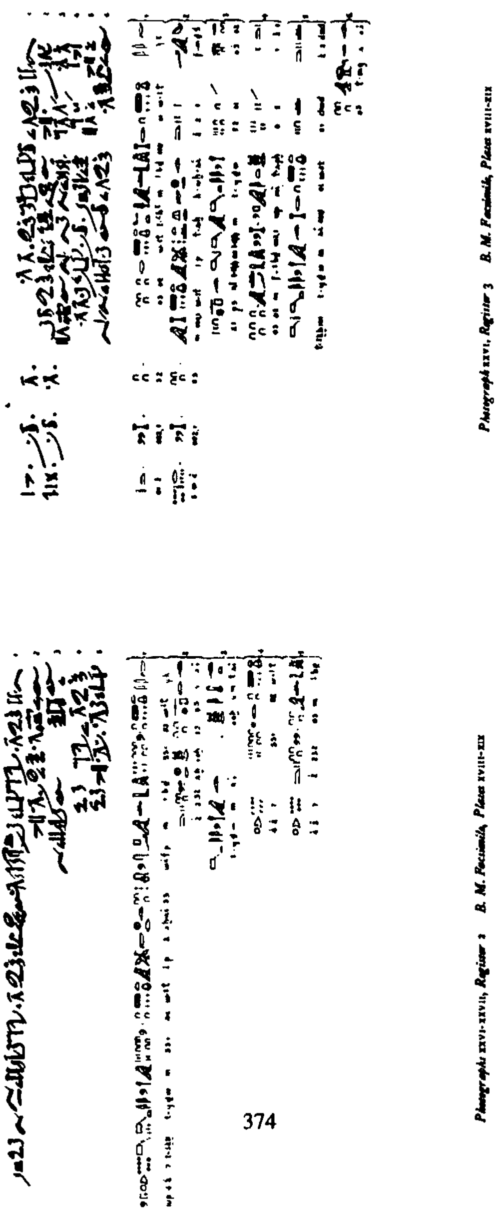
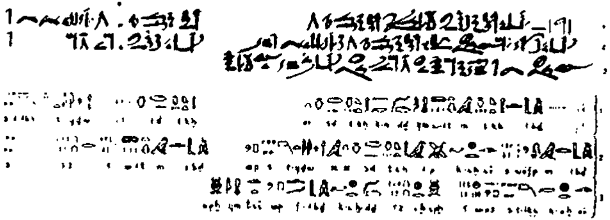
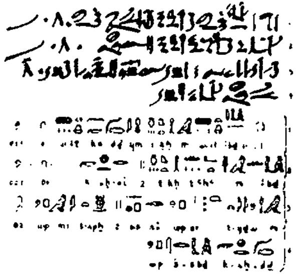


Fig. IV.2ww Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 97-98.

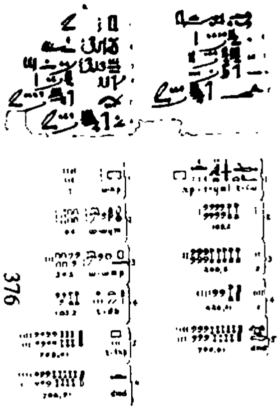


Photographs xxvi-xxvii, *Register* 4   *B. M. Facsimile, Plates* xviii-xix

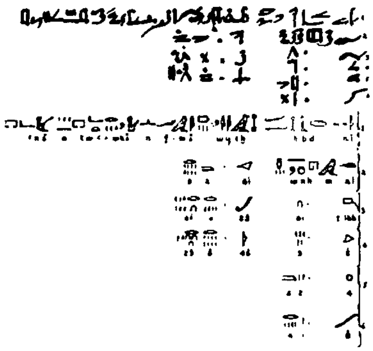


Photograph xxvi, *Register* 5   *B. M. Facsimile, Plates* xviii-xix

Fig. IV.2xx Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 99-100.



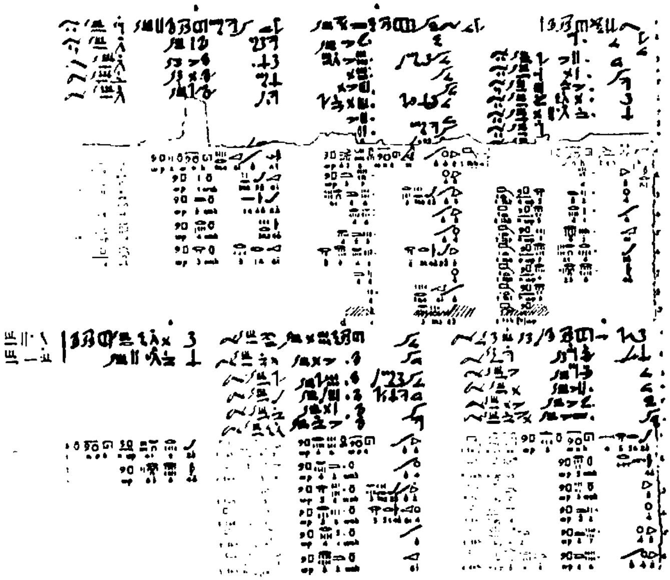
Photographs XXV-XXVI, *Register* 6   *B. M. Facsimile, Plate* XVIII



Photographs XXVI-XXVII, *Register* 5   *B. M. Facsimile, Plate* XIX

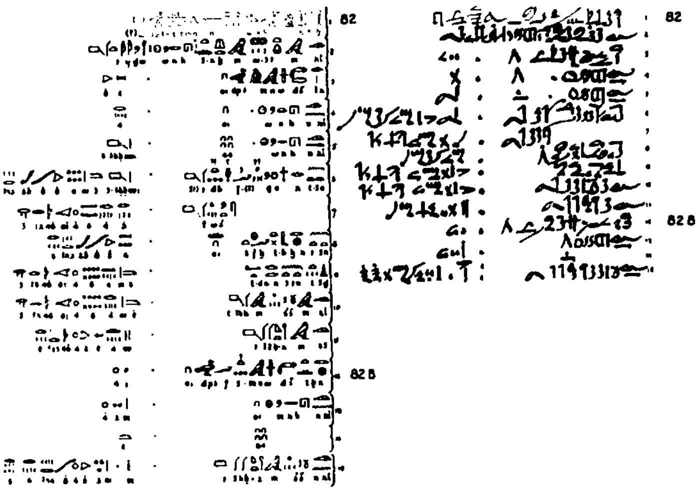
Fig. IV.2yy Hieratic text and Hieroglyphic transcription of the Rhind Mathematical Papyrus from Chace et al., *The Rhind Mathematical Papyrus*, Vol. 2 (Oberlin, Ohio, 1929), Plates 101-102.

PROBLEM 81



Photographs xxvi–xxviii, *Register* 6   *B. M. Facsimile, Plates* xviii–xx

PROBLEMS 82 AND 82B



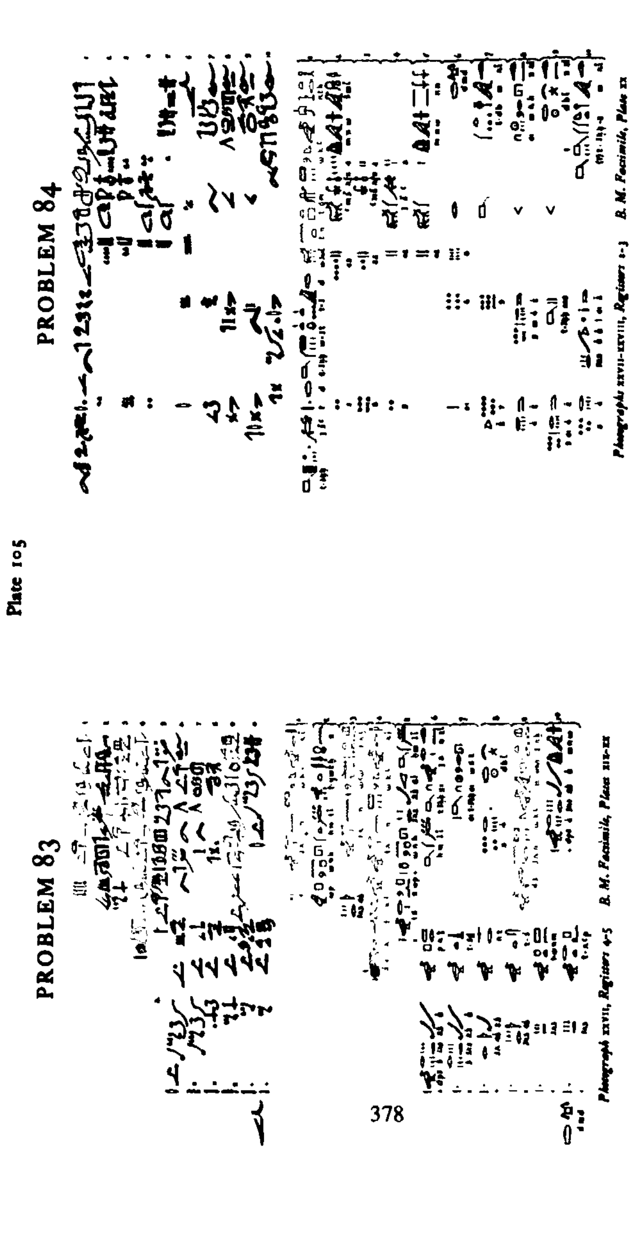


Plate 105
PROBLEM 84
PROBLEM 83
Photographs XXVII–XXVIII, Registers 1–3
B. M. Facsimile, Plate XX
Photograph XXVII, Registers 4–5
B. M. Facsimile, Plates XIX–XX

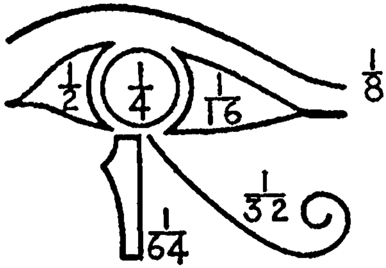


Fig. IV.3 Hieratic signs for the Horus-eye fractions of a heqat. Taken from Gardiner, *Egyptian Grammar*, 3rd edit., p. 197.

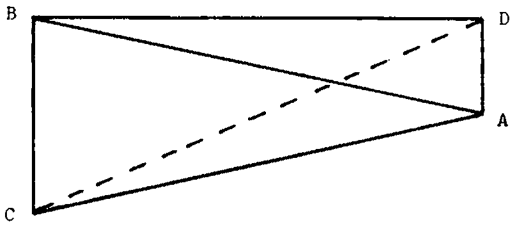


Fig. IV.4a Diagram for the Gunn and Peet interpretation of the term *meryt* (BD in this figure) as "quay" Taken from Gunn and Peet, "Four Geometrical Problems from the Moscow Mathematical Papyrus," p. 173.

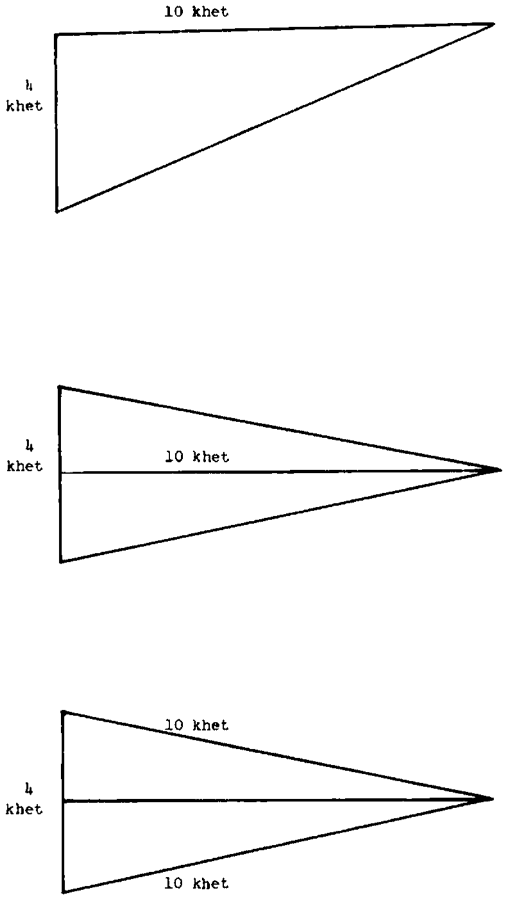


Fig. IV.4b Three possible interpretations of the triangular area calculated in Problem 51 of the Rhind papyrus. The uppermost is a right triangle with the *meryt* (10 khet) as the altitude. The middle figure is an isosceles triangle in which the *meryt* is once more the altitude. The bottom is also an isosceles triangle but with the *meryt* as one of the equal sides.

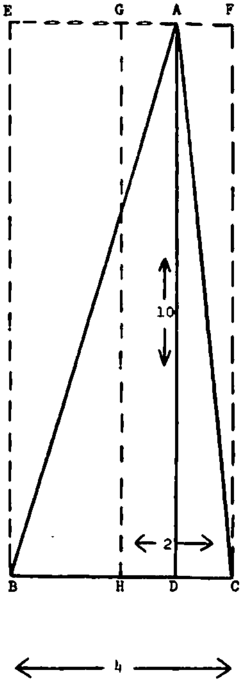
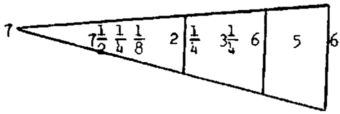


Fig. IV.4c A possible graphic solution of the area of a scalene triangle, marked with the numbers for the base (4) and the height (10) given in Problems 7 and 17 in Document IV 2



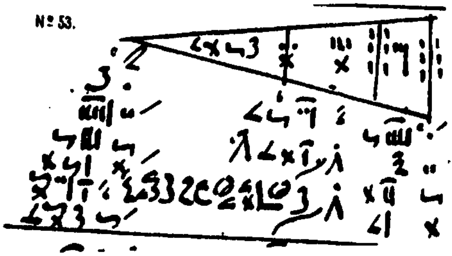


Fig. IV.5a Text and Figure for Problem 53 of the Rhind Mathematical Papyrus with the figure recopied and its hieratic numerals replaced by modern numerals.

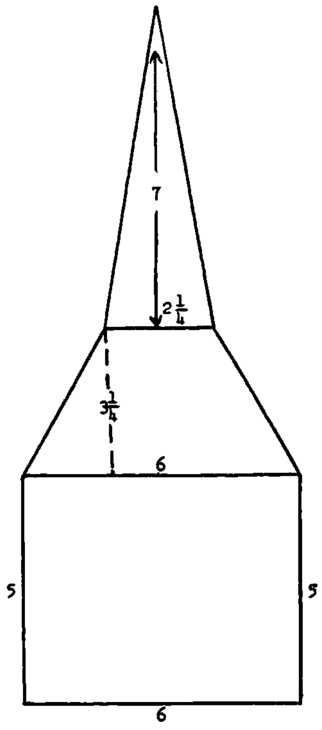


Fig. IV.5b A three-tiered figure reflecting the numbers given on the triangle appearing in the text in Fig.IV.5a. It thus is composed of a rectangle, a trapezoid, and a triangle.

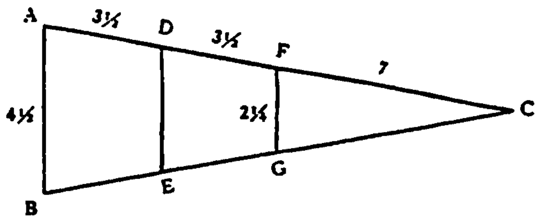
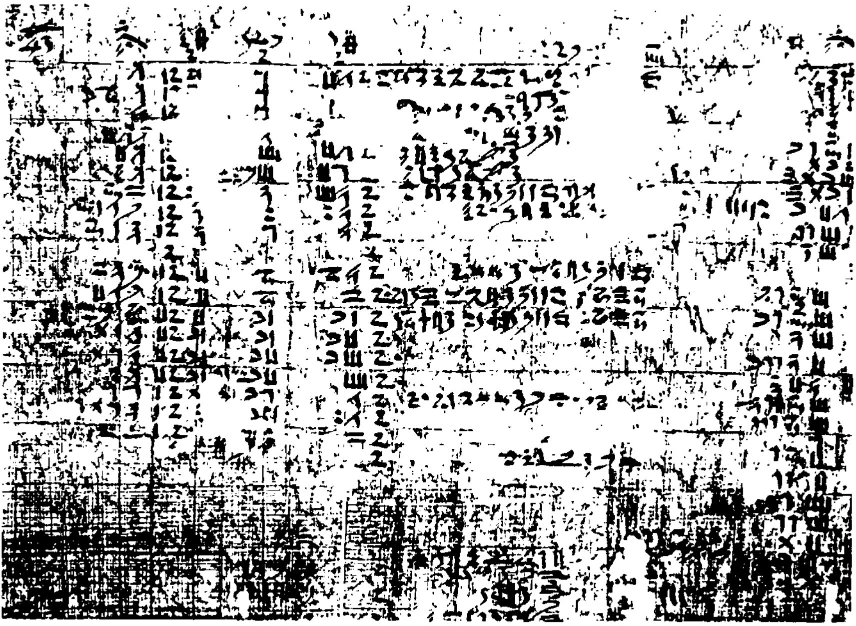


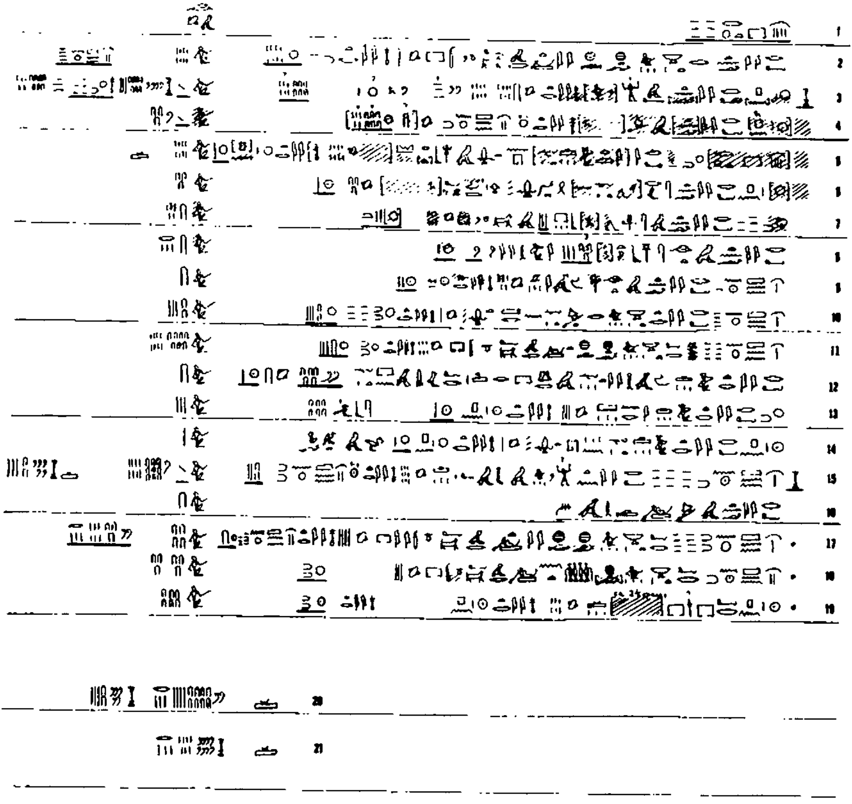
Fig. IV.5e Figure for Chace's interpretation of Problem 53 in his translation of the Rhind Papyrus, *op. cit.*, Vol. I, p. 94.

The Illustrations continue on the next page.



Section I

Fig. IV.18e Hieratic text of Section I of Papyrus Reisner I. Given as Plate 15 in W. Kelly Simpson, *ibid.*.



Section J

Fig. IV.18h Hieroglyphic transcription of Section J of Papyrus Reisner I. Given as Plate 16A in W. Kelly Simpson, *Ibid.*

Line	l.	b.	d.	Units	Volume	Enlistees	Detail
G 5	3	2	2	1	12	1 $\bar{5}$	
G 6, H 32	8	5	$\bar{4}$	1	10	1	Eastern Chapel.
G 14	8	3	$\bar{3}$	1	8	$\bar{2}$ $\bar{4}$ $\overline{20}$	
G 15	6	4	2	1	48	4 $\bar{2}$ $\bar{4}$ $\overline{20}$	
G 16	4	2	2	1	16	1 $\bar{2}$ $\overline{10}$	
G 17, H 33	4	4	2	2	64	6 $\bar{4}$ $\overline{10}$ $\overline{20}$	Footings.
G 18, H 34	3	3	2	2	36	3 $\bar{2}$ $\overline{10}$	Footings.
H 31	15	5	$\bar{4}$	1	18 $\bar{2}$ $\bar{4}$		Great Chamber.
H 7	3 $\bar{4}$	1 $\bar{2}$	$\bar{2}$	2	4 $\bar{2}$ $\bar{4}$ $\bar{8}$		
H 8	2 $\bar{2}$	1 $\bar{2}$	$\bar{2}$	1	1 $\bar{2}$ $\bar{4}$ $\bar{8}$		
H 9	4 $\bar{2}$	1 $\bar{2}$	1	2	13 $\bar{2}$		
H 11	3c 1p	1	1	1	3c 1p		
H 17	4	1c 3p	1	4	22c 6p		
H 25	1 $\bar{2}$	$\bar{2}$ $\bar{4}$	1	2	2 $\bar{4}$		
H 26	2 $\bar{2}$	$\bar{2}$ $\bar{4}$	1	2	3 $\bar{2}$ $\bar{4}$		
H 27	3 $\bar{2}$	1 $\bar{2}$	1 $\bar{2}$	2	15 $\bar{2}$ $\bar{4}$		
H 30	12	5	$\bar{4}$	1	15		Great Chamber.
I 2	12	5	$\bar{2}$	1	30		Great Chamber.
I 3	15	5	$\bar{2}$	1	37 $\bar{2}$		August Chamber.
I 4	8	5	$\bar{2}$	1	20		Eastern Chapel.
I 5	18	11	$\bar{3}$	1	132		
I 6	32	4	$\bar{4}$	1	32		Western.
I 7, G 10	52	3	$\bar{4}$	1	39		Eastern.
I 8	24	5p	$\bar{2}$	1	8c 4p		
I 9	26	2	5p	1	111c 3p		Carrying srft.
I 10	20	5	5p	1	71c 3p		Carrying srft.
I 12	27	7	2	1	378		Loosening brick clay
I 13	8	7	2	1	112		Water from a field.
I 14	1 $\bar{2}$	1 $\bar{2}$	2	2	9		For tower.
I 15	2 $\bar{2}$	1 $\bar{2}$	1 $\bar{2}$	2	11 $\bar{4}$		For tower.
I 20	8	6	1	1	48		

Fig. IV.19 Correct calculations of volumes by the scribe of Reisner Papyrus I. Taken from Gillings. *Mathematics in the Time of the Pharaohs*, Table 22.2, p. 221.

Line	l.	b.	d.	Units	Volume	Enlistees	Detail
G 8	35	11	$\bar{2}$	1	192 $\bar{2}$	19 $\bar{2}$ for 19 $\bar{4}$	
H 35	8	5 for 9	$\bar{4}$	1	18		
I 16	3 $\bar{2}$	2 $\bar{2}$	1 $\bar{2}$	2	25 $\bar{4}$ for 26 $\bar{4}$		For the tower.
I 17	4	2 $\bar{2}$	1 $\bar{2}$	2	36 for 30		For the tower.
I 18	10	5 $\bar{2}$	$\bar{4}$ for 1	1	55		Brick clay.

B. Errors, Misreadings, or Approximations.

Line	l.	b.	d.	Units	Volume	Enlistees
G 7	3	2 $\bar{2}$	4	1	2 $\bar{2}$ for 1 $\bar{2}$ $\bar{4}$ $\bar{8}$	5 $\overline{20}$ correct for 2 $\bar{2}$.
G 9	13	11	1 $\bar{2}$	1	214 $\bar{2}$	21 $\bar{2}$ for 21 $\bar{4}$ $\bar{5}$.
G 10	52	3	$\bar{4}$	1	39	4 for 3($\bar{2}$ $\bar{3}$ $\overline{15}$).
G 11	32	4	$\bar{2}$	1	85 for 64	8 $\bar{2}$ correct for 85.
G 12	3 $\bar{2}$	2	$\bar{3}$	1	4 $\bar{3}$	$\bar{2}$ for ($\bar{3}$ $\overline{10}$ $\overline{30}$).
G 13	10 $\bar{2}$	8 $\bar{2}$	$\bar{5}$	1	27 for 29 $\bar{2}$ $\bar{4}$	2 $\bar{2}$ $\bar{5}$ correct for 27.
H 20	3c 3p	1c 3p	1	1	5c	5c for 4c 6p ($\bar{4}$ $\overline{28}$)p.

C. Minor Errors.

Line	l.	b.	d.	Units	Volume
H 10	4c 1p	1 $\bar{2}$	$\bar{2}$	1	(3c 2f) for (3c 3f).
H 13	2c 3p	2c 3p	$\bar{3}$	1	(3c 6p 1 $\bar{3}$f) for (3c 6p 2f)($\overline{14}$ $\overline{42}$f).
H 14	2c 2f	1 $\bar{2}$	1c 1p 1f	1	(3c 4p 1 $\bar{2}$f) for (3c 4p 2f)($\bar{2}$ $\overline{28}$f).
H 15	1c 5p	1 $\bar{2}$	5p	2	(3c 5p) for (3c 4p 2f)($\bar{2}$ $\bar{4}$ $\overline{14}$ $\overline{28}$f).
H 18	3c 2p	1c 2p	6p	1	(3c 3p 2 $\bar{3}$f) for (3c 4p 1 $\bar{3}$f)($\overline{15}$f).
H 22	1 $\bar{3}$	1c 3p	1	1	(2c 2p 2f) for (2c 2p 2 $\bar{3}$f).
H 24	3c 5p	1c 2p	6p	1	(4c 2f) for (4c 2 $\bar{3}$f)(5 $\overline{15}$) approx.
H 28	4c 4p	1c 5p	1c 2p	2	(20c 1p 1 $\bar{2}$f) for (20c 1p $\bar{4}$f) approx.

D. Major Errors.

Line	l.	b.	d.	Units	Volume
H 16	2c 3p	1c 4p	5p 2f	1	(2c 5p 2 $\bar{2}$f) for (2c 6p 3 $\bar{2}$f) approx.
H 19	3c 5p 2f	1c 3p	1	1	(4c 2p 3f) for (5c 2p 3 $\bar{4}$f)($\bar{7}$ $\overline{28}$).
H 21	1c 5p	1c 3p	1	1	(2c 4p 1 . . . f) for (2c 3p $\bar{2}$ $\overline{14}$f).
H 23	4	1c 6p	6p	1	(4c 1f) for (6c 2p 2 $\bar{4}$f)28.

E. Possible Restorations.

Line	l.	b.	d.	Units	Volume	Restoration
H 2	2c 5p	6p	[ ]	2	4c 1 $\bar{3}$f	d = 6p 1f (probably too great).
H 3	2 $\bar{4}$	6p	[ ]	1	1c 2p 2 . . . f	d = 5p (gives v = 1c 2p 2 $\bar{2}$ $\overline{14}$f).
H 4	2 $\bar{4}$	6p	[ ]	[ ]	1 . . . 2p 2 . . . f	d = 5p (units = 1, as above).
H 5	2 $\bar{4}$	6p	[ ]	1	1c 3p 1 $\bar{2}$f	d = 5p 2f (probably too great)
H 6	4 $\bar{4}$	[ ]	[ ]	2	6 $\bar{2}$ $\bar{4}$	b = 6p

Fig. IV.20 Calculations with errors by the scribe of Reisner Papyrus I. Taken from Gillings, *op. cit.*, Table 22.3, pp. 222-23.

4 fingers =	1 palm					
7 palms =	1 cubit					
1p =	$\bar{7}$					c
2p =	$\bar{4}$	$\bar{28}$				c
3p =	$\bar{4}$	$\bar{7}$	$\bar{28}$			c
4p =	$\bar{2}$	$\bar{14}$				c
5p =	$\bar{2}$	$\bar{7}$	$\bar{14}$			c
6p =	$\bar{2}$	$\bar{4}$	$\bar{14}$	$\bar{28}$		c

Fig. IV.21. Table of fingers, palms, and cubits. Taken from Gillings, *op. cit.*, Table 22.5, p. 225.

length	breadth	depth	units	volume
3 c 5 p	1 c 2 p	6 p	1	4 c 2 f.

Then,

	c	p	c	p		
			3	5		
\1			3	5		
\		2		6	10	
Totals	1	2	4	5	3.	
			4	5	3	
\		6		24	30	18
Totals		6	4	0	4	4.

Fig. IV.22 An Egyptian technique for calculating the volume in Section II, line 25 of the Reisner Papyrus I suggested by Gillings, *op. cit.*, p. 227.

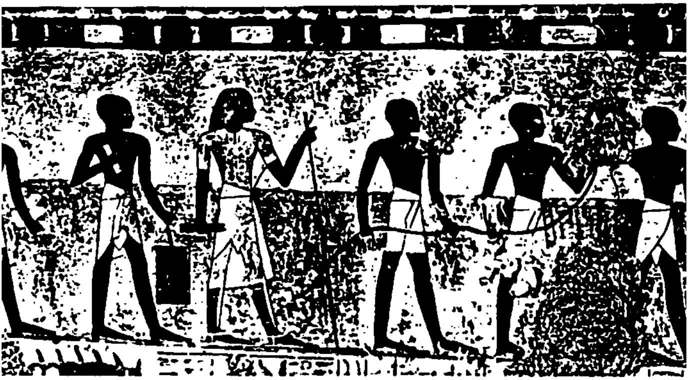


Fig. IV.23 Field hands carrying a measuring cord, a painting on stucco from the tomb of Djeserkaseneb in the reign of Tuthmosis IV (ca. 1401–1391 B.C.). Taken from W. Wreszinski, *Atlas zur altägyptischen Kulturgeschichte* (Leipzig, 1923), Tafel 2

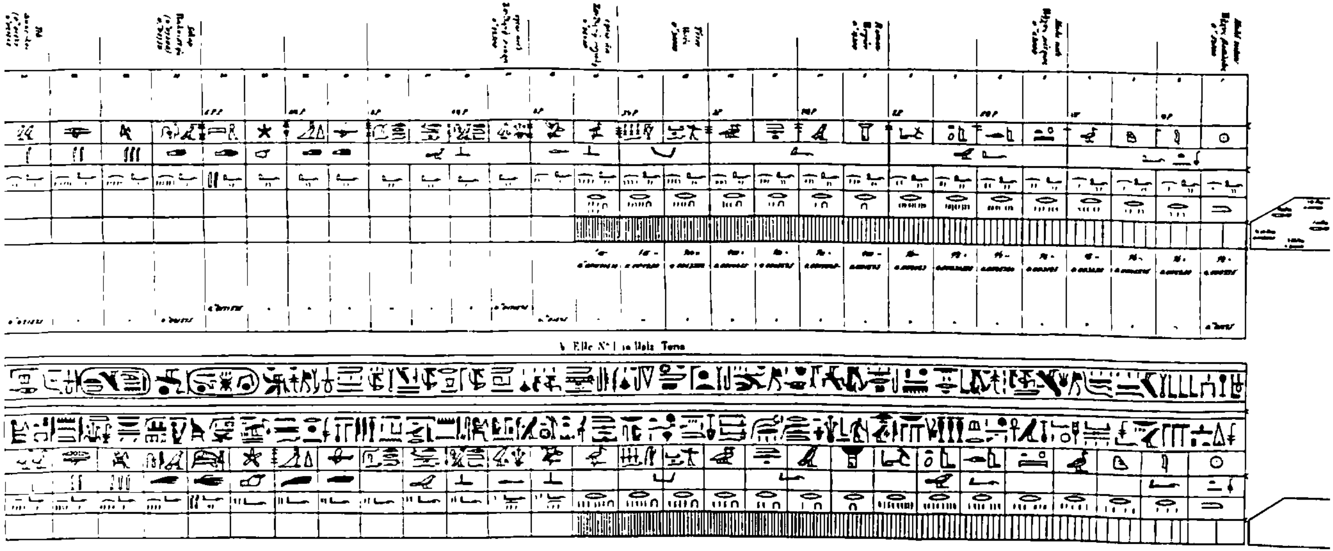


Fig. IV.24 (Tafel I) (a) Reconstruction of an Ancient Egyptian cubit-rod. (b) Cubit-rod no. I, a wooden cubit-rod at Turin. Taken from the Tafeln appended to Richard Lepsius, *Die altägyptische Elle und ihre Entheilung.*

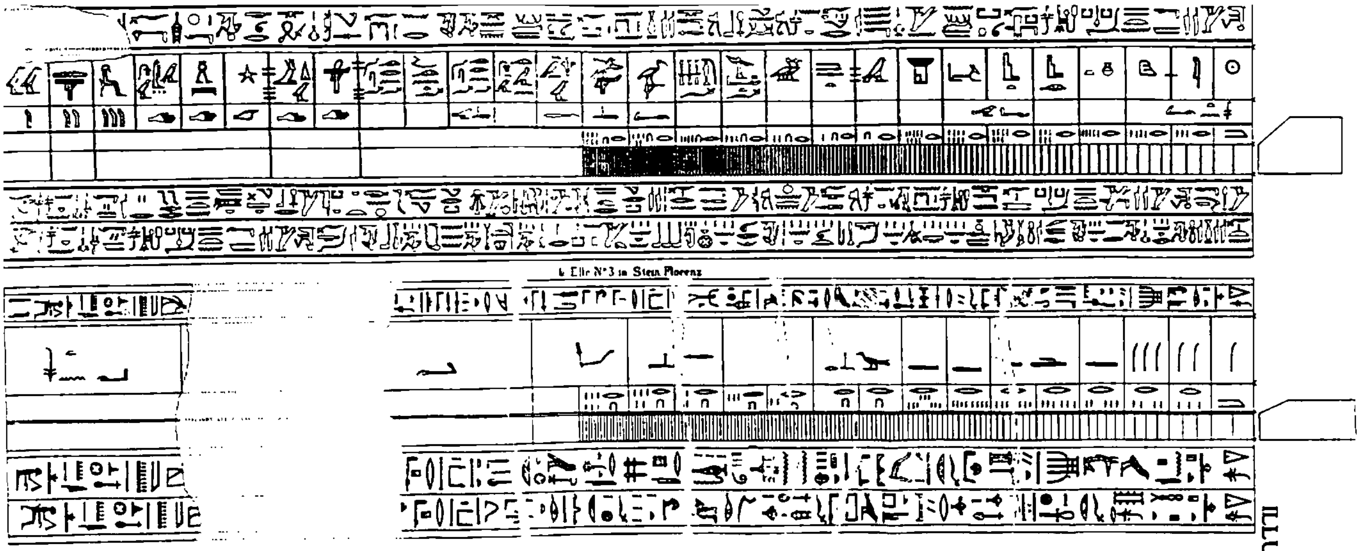


Fig. IV.24 (Tafel 2) (a) Cubit-rod no. 2, a wooden rod in Paris. (b) Cubit-rod no. 3, a stone rod from Florence. Taken from the Tafeln

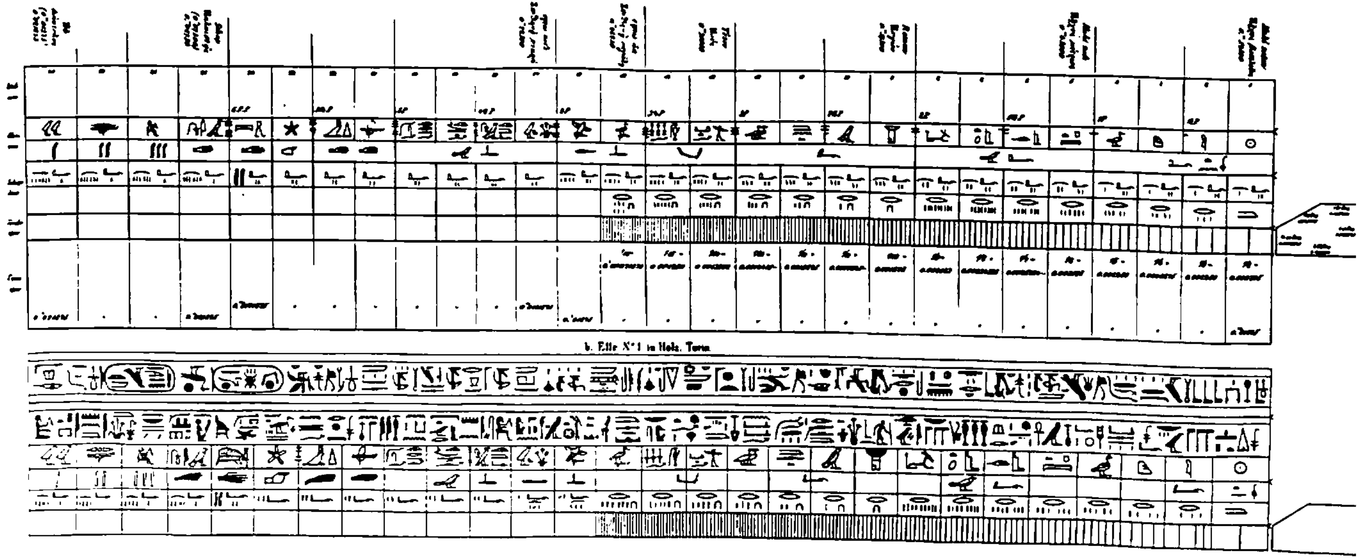


Fig. IV.24 (Tafel I) (a) Reconstruction of an Ancient Egyptian cubit-rod. (b) Cubit-rod no. I, a wooden cubit-rod at Turin. Taken from the Tafeln appended to Richard Lepsius, *Die altägyptische Elle und ihre Entheilung.*

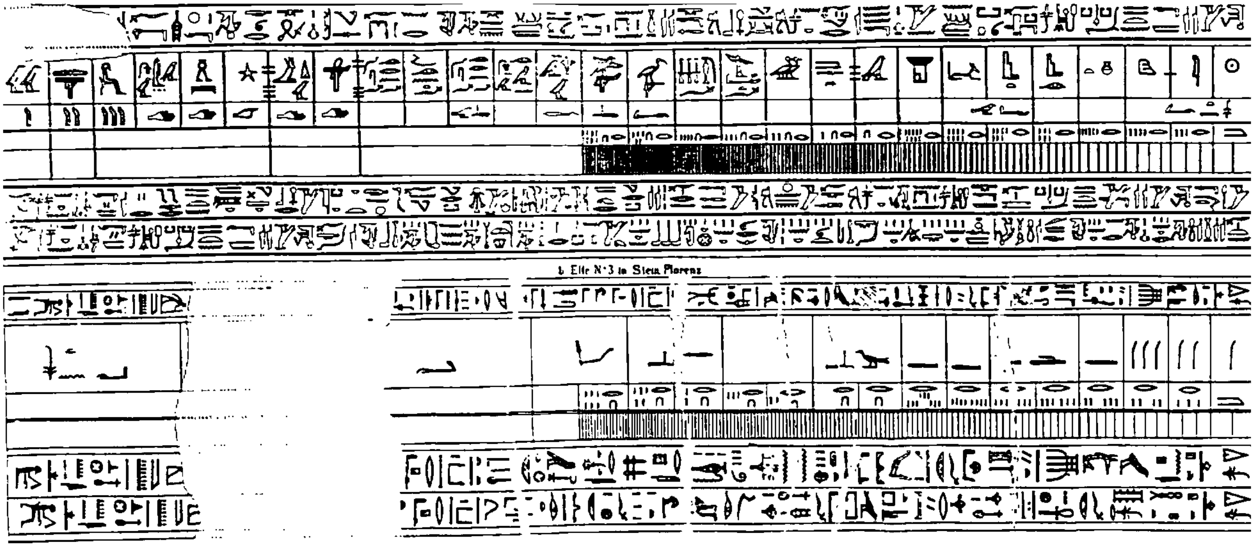


Fig. IV.24 (Tafel 2) (a) Cubit-rod no. 2, a wooden rod in Paris. (b) Cubit-rod no. 3, a stone rod from Florence. Taken from the Tafeln

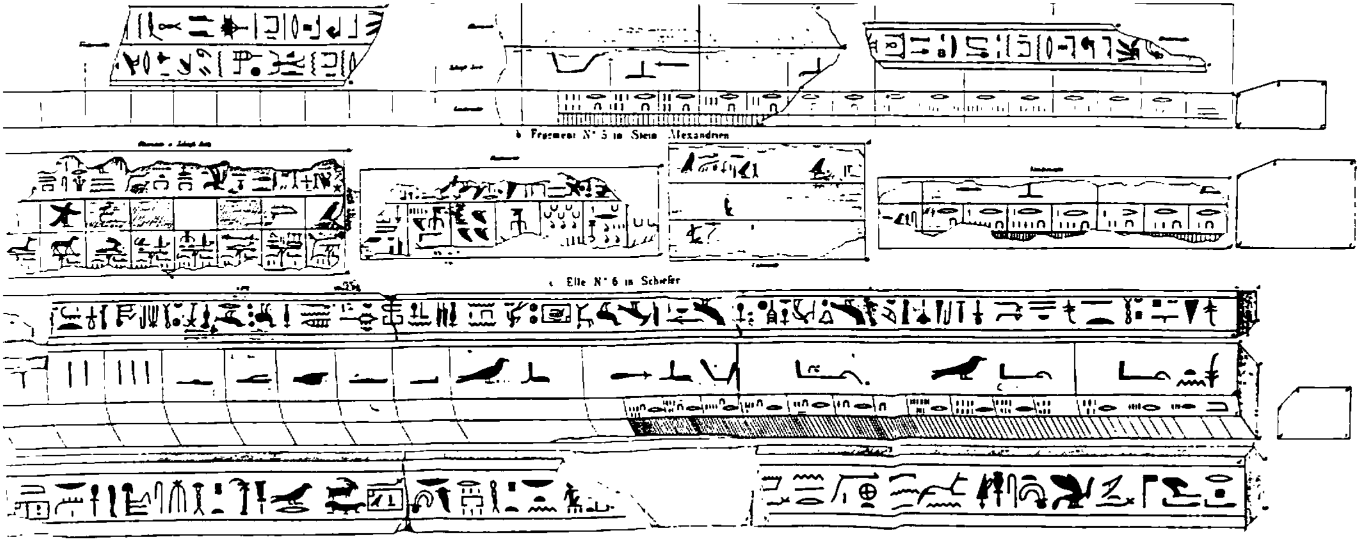


Fig. IV.24 (Tafel 3) (a) Fragment no. 4 of slate. (b) Fragment no. 5 of stone in Alexandria. (c) Cubit-rod no. 6 of slate. Taken from the Tafeln appended to Richard Lepsius. *Die altägyptische Elle und ihre Entheilung.*

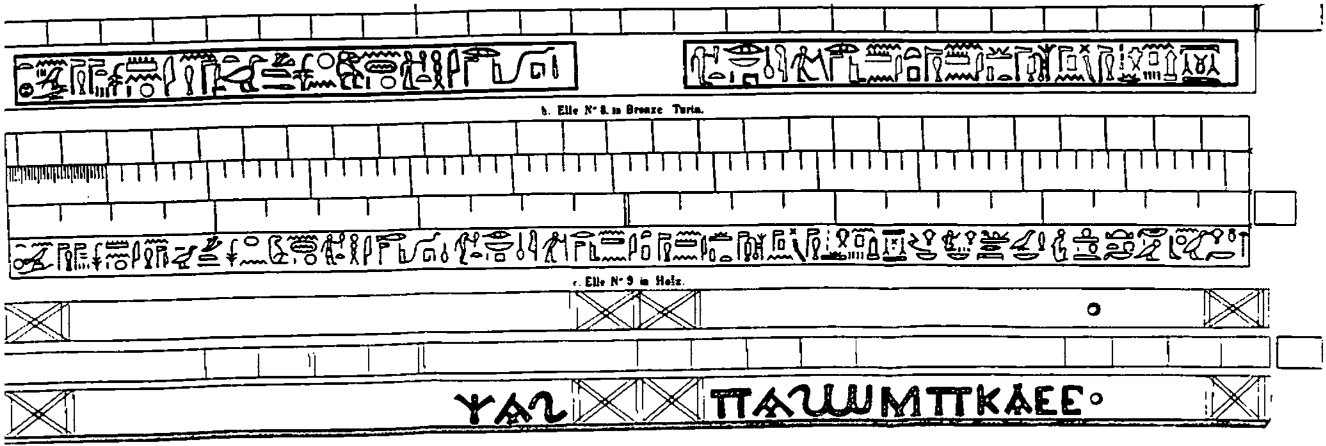


Fig. IV.24 (Tafel 4) (a) Cubit-rod no. 7 of basalt in Turin. (b) Cubit-rod no. 8 of bronze in Turin. (c) Cubit rod of wood. Taken from the Tafeln appended to Richard Lepsius. *Die altägyptische Elle und ihre Entheilung.*

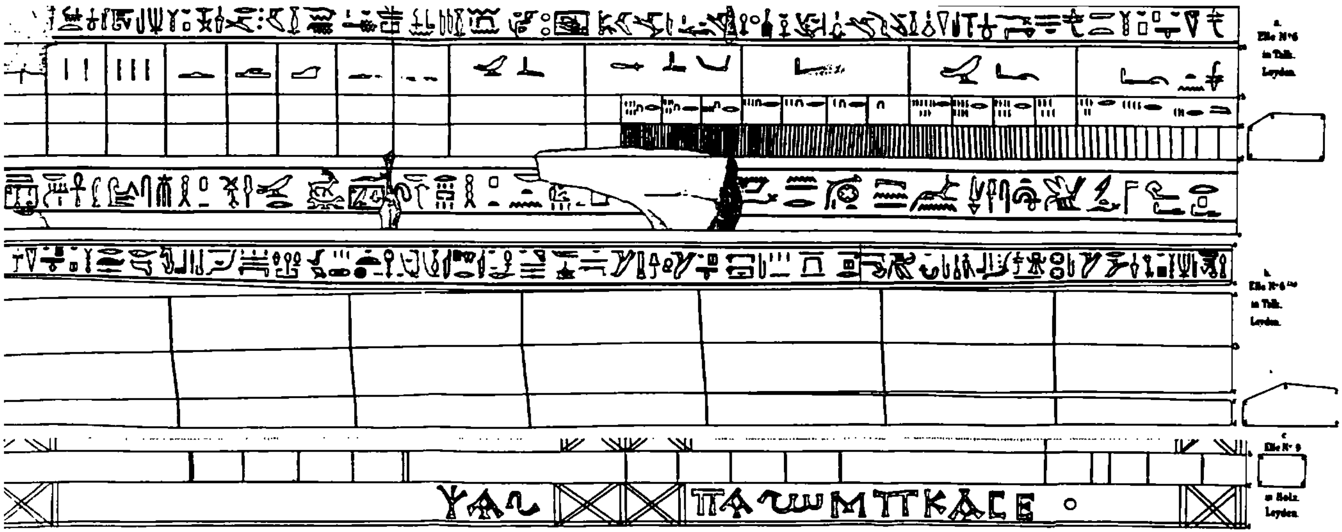


Fig. IV.24 (Tafel 5) (a) Cubit-rod no. 6, reproduced in talc, in Leiden. (b) Cubit-rod no. 6bis, in talc, in Leiden. (c) Cubit-rod no. 9

Fig. IV.25 Photograph of a cubit-rod in the Musée du Louvre, Paris (No. 1538). See the clearer reproduction of this given by Lepsius in my Fig. IV.24 (Tafel 2). Taken from Museum of Fine Arts, Boston, *Egypt's Golden Age: The Art of Living in the New Kingdom 1558-1085 B.C.*, Photo. no. 30, p. 59.

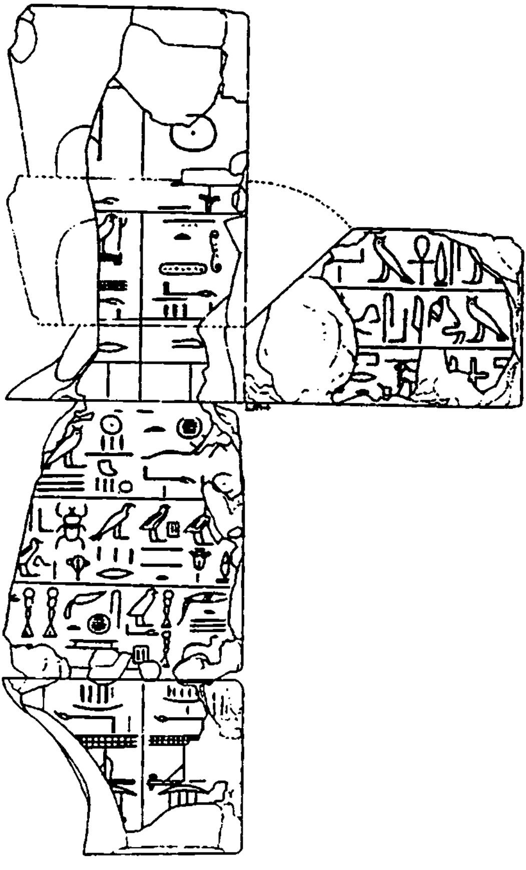
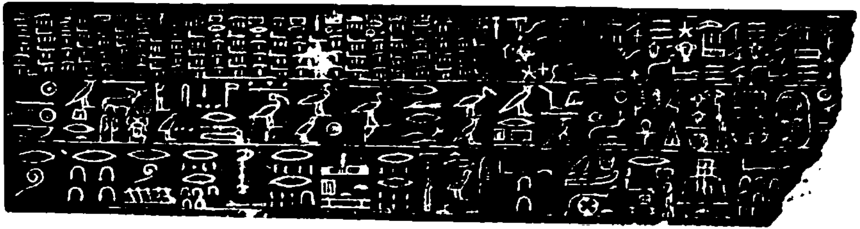
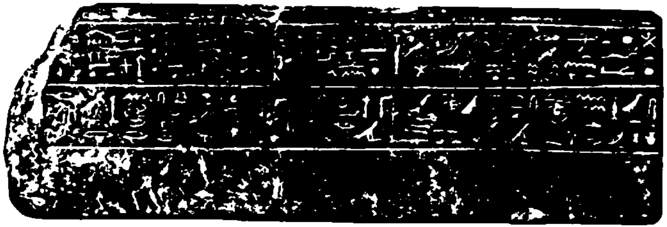


Fig. IV.26 Fragment of a ceremonial cubit-rod in the Metropolitan Museum of Art, N.Y., made from chert. End view and orthographic projection. Length of fragment 1 13/16 in. Taken from W.C. Hayes, *The Scepter of Egypt*, Part II (Greenwich, Conn., 1959), Fig. 263, p. 413.

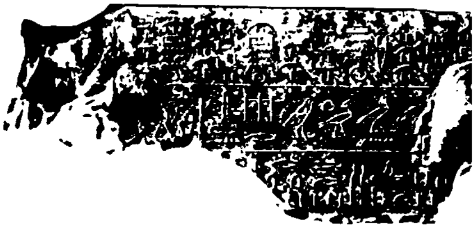
1. Aus Karnak. Kairoer Museum.



2. Aus Karnak. Kairoer Museum.

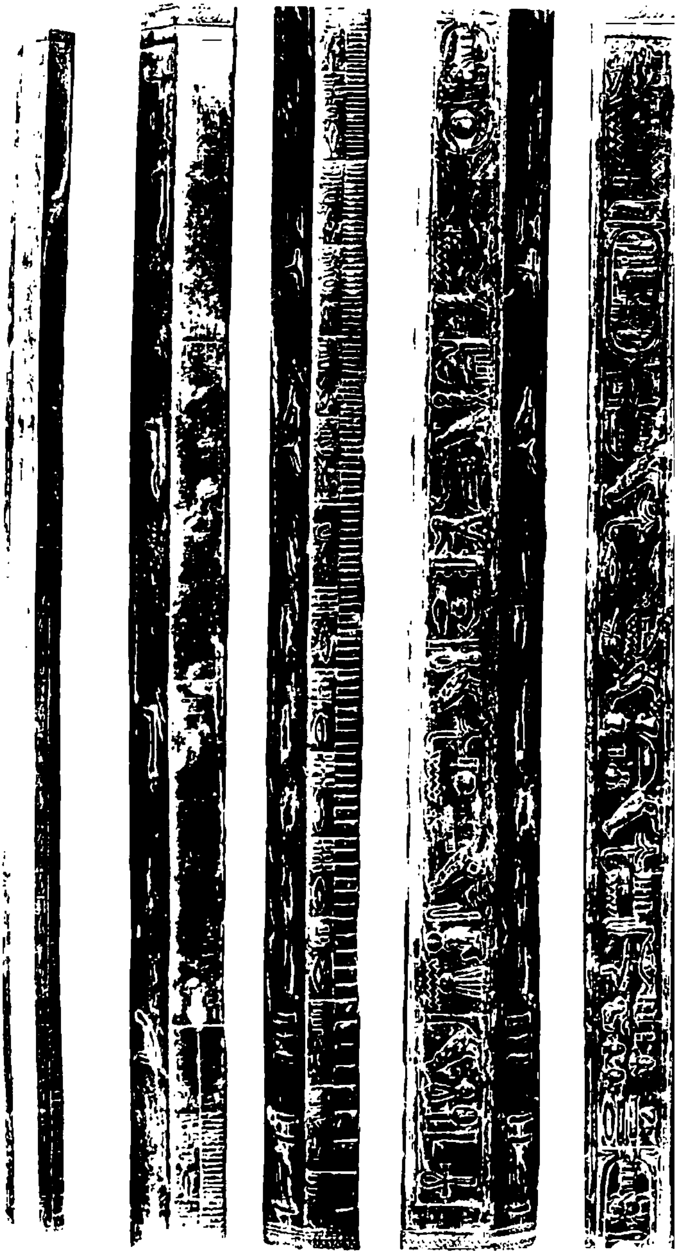


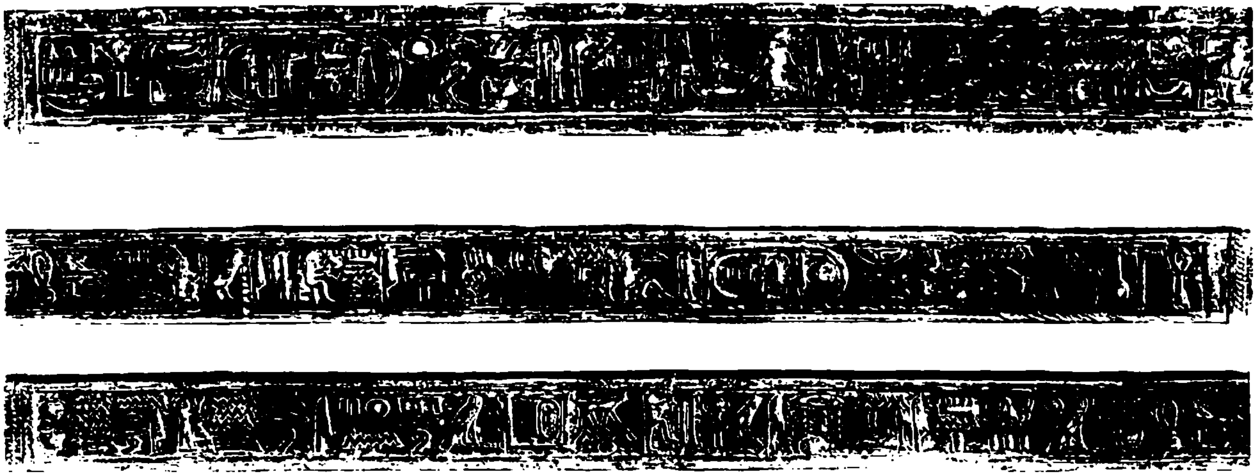
3. Aus Sais. Kairoer Museum. Gen.-Kat. Nr. 1942.



4. Berliner Museum Nr. 7358.

Fig. IV.27a Ceremonial cubit-rods containing tables that might refer to water-clocks and shadow-clocks. Taken from L. Borchardt, *Die altägyptische Zeitmessung*, Tafel II.





(Fig. IV.27c)

Figs. IV.27b and IV.27c The Golden Cubit-rod from the tomb of Cha in the Necropolis of Thebes. Taken from Schiaparelli. *La tomba intatta del architteto Cha nella necropoli di Tebe* (Torino. 1927). Figs. 155 and 156.

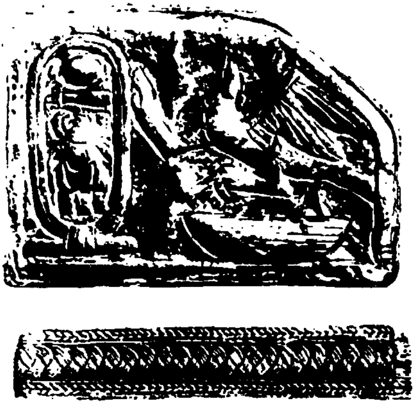


Fig. IV.27d Enlargements of one of the heads and the frieze of the Cubit-rod pictured in Figs. IV.27b and IV.27c. Taken from Schiaparelli, *ibid.*, Fig. 154.

Fig. IV.27e A working, hinged cubit rod of Acacia wood, also from the tomb of Cha. Taken from Schiaparelli, *ibid.*, Fig. 47.

+ 5	1 2/3 · 1/3	1/3 + 1/15
7	1 3/4 · 1/4	1/4 + 1/28
9	1 1/2 · 1/2	1/6 + 1/18
11	1 2/3 + 1/6 · 1/6	1/6 + 1/66
13	1 1/2 + 1/8 · 1/4 · 1/8	1/8 + 1/52 + 1/104
15	1 1/2 · 1/2	1/10 + 1/30
17	1 1/3 + 1/12 · 1/3 · 1/4	1/12 + 1/51 + 1/68
19	1 1/2 + 1/12 · 1/4 · 1/6	1/12 + 1/76 + 1/114
21	1 1/2 + 1/2	1/14 + 1/42
23	1 2/3 + 1/4 · 1/12	1/12 + 1/276
25	1 2/3 · 1/3	1/15 + 1/75
27	1 1/2 · 1/2	1/18 + 1/54
29	1 1/6 + 1/24 · 1/2 · 1/6 · 1/8	1/24 + 1/58 + 1/174 + 1/232
31	1 1/2 + 1/20 · 1/4 · 1/5	1/20 + 1/124 + 1/155
33	1 1/2 · 1/2	1/22 + 1/66
35	1 1/6 · 2/3 + 1/6	1/30 + 1/42
37	1 1/2 + 1/24 · 1/3 · 1/8	1/24 + 1/111 + 1/296
39	1 1/2 · 1/2	1/26 + 1/78
41	1 2/3 + 1/24 · 1/6 · 1/8	1/24 + 1/246 + 1/328
43	1 1/42 · 1/2 · 1/3 · 1/7	1/42 + 1/86 + 1/129 + 1/301
45	1 1/2 · 1/2	1/30 + 1/90
47	1 1/2 + 1/15 · 1/3 · 1/10	1/30 + 1/141 + 1/470
49	1 1/2 + 1/4 · 1/4	1/28 + 1/196
51	1 1/2 · 1/2	1/34 + 1/102
53	1 2/3 + 1/10 · 1/6 · 1/15	1/30 + 1/318 + 1/795
55	1 2/3 + 1/6 · 1/6	1/30 + 1/330
57	1 1/2 · 1/2	1/38 + 1/114
59	1 1/2 + 1/12 + 1/18 · 1/4 · 1/9	1/36 + 1/236 + 1/531
61	1 1/2 + 1/40 · 1/4 · 1/8 · 1/10	1/40 + 1/244 + 1/488 + 1/610
63	1 1/2 · 1/2	1/42 + 1/126
65	1 2/3 · 1/3	1/39 + 1/195
67	1 1/2 + 1/8 + 1/20 · 1/5 · 1/8	1/40 + 1/335 + 1/536
69	1 1/2 · 1/2	1/46 + 1/138
71	1 1/2 + 1/4 + 1/40 · 1/8 · 1/10	1/40 + 1/568 + 1/710
73	1 1/6 + 1/20 · 1/3 · 1/4 · 1/5	1/60 + 1/219 + 1/292 + 1/365
75	1 1/2 · 1/2	1/50 + 1/150
77	1 1/2 + 1/4 · 1/4	1/44 + 1/308
79	1 1/4 + 1/15 · 1/3 · 1/4 · 1/10	1/60 + 1/237 + 1/316 + 1/790
81	1 1/2 · 1/2	1/54 + 1/162
83	1 1/3 + 1/20 · 1/4 · 1/5 · 1/6	1/60 + 1/332 + 1/415 + 1/498
85	1 2/3 · 1/3	1/51 + 1/255
87	1 1/2 · 1/2	1/58 + 1/174
89	1 1/3 + 1/10 + 1/20 · 1/4 · 1/6 · 1/10	1/60 + 1/356 + 1/534 + 1/890
91	1 1/5 + 1/10 · 2/3 + 1/30	1/70 + 1/130
93	1 1/2 · 1/2	1/62 + 1/186
95	1 1/2 + 1/12 · 1/4 · 1/6	1/60 + 1/380 + 1/570
97	1 1/2 + 1/8 + 1/14 + 1/28 · 1/7 · 1/8	1/56 + 1/679 + 1/776
99	1 1/2 · 1/2	1/66 + 1/198
101	1 · 1/2 · 1/3 · 1/6	1/101 + 1/202 + 1/303 + 1/606

Fig. IV.28 The "Table of Two" from the Rhind Mathematical Papyrus as tabulated by E. Peet, *The Rhind Mathematical Papyrus* (London, 1923), p. 37.

DIVISION OF 2 BY ODD NUMBERS

	Number	First Multiplier	Corresponding Product	Remainder	Answer
A	3	2/3	2		2/3
A	5	1/3	1 2/3	1/3	1/3 1/15
B	7	1/4	1 1/2 1/4	1/4	1/4 1/28
A	9	1/6	1 1/2	1/2	1/6 1/18
A	11	1/6	1 2/3 1/6	1/6	1/6 1/66
B	13	1/8	1 1/2 1/8	1/4 1/8	1/8 1/52 1/104
BD	15	1/10	1 1/2	1/2	1/10 1/30
A	17	1/12	1 1/4 1/6 or 1 1/3 1/12	1/3 1/4	1/12 1/51 1/68
A	19	1/12	1 1/2 1/12	1/4 1/6	1/12 1/76 1/114
C	21	1/14	1 1/2	1/2	1/14 1/42
A	23	1/12	1 1/2 1/4 1/6 or 1 2/3 1/4	1/12	1/12 1/276
AD	25	1/15	1 2/3	1/3	1/15 1/75
C	27	1/18	1 1/2	1/2	1/18 1/54
A	29	1/24	1 1/6 1/24	1/2 1/6 1/8	1/24 1/58 1/174 1/232
BD	31	1/20	1 1/2 1/20	1/4 1/5	1/20 1/124 1/155
C	33	1/22	1 1/2	1/2	1/22 1/66
E	35	1/30 1/42	1 1/6 2/3 1/6		1/30 1/42
A	37	1/24	1 1/2 1/24	1/3 1/8	1/24 1/111 1/296
C	39	1/26	1 1/2	1/2	1/26 1/78
A	41	1/24	1 2/3 1/24	1/6 1/8	1/24 1/246 1/328
AD	43	1/42	1 1/42	1/2 1/3 1/7	1/42 1/86 1/129 1/301
C	45	1/30	1 1/2	1/2	1/30 1/90
AD	47	1/30	1 1/2 1/15	1/3 1/10	1/30 1/141 1/470
BD	49	1/28	1 1/2 1/4	1/4	1/28 1/196
C	51	1/34	1 1/2	1/2	1/34 1/102
AD	53	1/30	1 2/3 1/10	1/6 1/15	1/30 1/318 1/795
AD	55	1/30	1 2/3 1/6	1/6	1/30 1/330
C	57	1/38	1 1/2	1/2	1/38 1/114
AD	59	1/36	1 1/2 1/12 1/18	1/4 1/9	1/36 1/236 1/531
BD	61	1/40	1 1/2 1/40	1/4 1/8 1/10	1/40 1/244 1/488 1/610
C	63	1/42	1 1/2	1/2	1/42 1/126
C	65	1/39	1 2/3	1/3	1/39 1/195
BD	67	1/40	1 1/2 1/8 1/20	1/5 1/8	1/40 1/335 1/536
C	69	1/46	1 1/2	1/2	1/46 1/138
BD	71	1/40	1 1/2 1/4 1/40	1/8 1/10	1/40 1/568 1/710
AD	73	1/60	1 1/6 1/20	1/2 1/4 1/30	1/60 1/219 1/292 1/365
C	75	1/50	1 1/2	1/2	1/50 1/150
C	77	1/44	1 1/2 1/4	1/4	1/44 1/308
AD	79	1/60	1 1/4 1/15	1/3 1/4 1/10	1/60 1/237 1/316 1/790
C	81	1/54	1 1/2	1/2	1/54 1/162
AD	83	1/60	1 1/3 1/20	1/4 1/5 1/6	1/60 1/332 1/415 1/498
C	85	1/51	1 2/3	1/3	1/51 1/255
C	87	1/58	1 1/2	1/2	1/58 1/174
AD	89	1/60	1 1/3 1/10 1/20	1/4 1/6 1/10	1/60 1/356 1/534 1/890
E	91	1/70 1/130	1 1/5 1/10 2/3 1/30		1/70 1/130
C	93	1/62	1 1/2	1/2	1/62 1/186
AD	95	1/60	1 1/2 1/12	1/4 1/6	1/60 1/380 1/570
BD	97	1/56	1 1/2 1/8 1/14 1/28	1/7 1/8	1/56 1/679 1/776
C	99	1/66	1 1/2	1/2	1/66 1/198
E	101	1/101	1	1/2 1/3 1/6	1/101 1/202 1/303 1/606

recto, page 1, colonne 1.

]	1/3 au nombre 4000	
de	un les 2/3 = 2/3	
de	2	1 1/3
de	3	2
de	4	2 2/3
de	5	3 1/3
de	6	4
de	7	4 2/3
de	8	5 1/3
de	9	6
de	10	6 2/3
de	20	13 1/3
de	30	20
de	40	26 2/3
de	50	33 1/3
de	60	40
de	70	46 2/3
de	80	53 1/3
de	90	60

τῶ[ν	Ρ]	ΞϚ𐅷	de	100	66 2/3
τῶ[ν	Σ]	ΡΛΓγ′	de	200	133 1/3
τῶ[ν	Τ]	Σ	de	300	200
τῶ[ν	Υ]	ΣΞϚ𐅷	de	400	266 2/3
τῶ[ν	Φ]	ΤΛΓγ′	de	500	333 1/3
τῶ[ν	Χ]	·Υ	de	600	400
τ[ῶν	Ψ]	ΥΞϚ𐅷	de	700	466 2/3
τῶν	Ω	ΦΛΓγ′	de	800	533 1/3
τῶν	ϡ	Χ	de	900	600
τῶν	͵Α	ΧΞϚ𐅷	de	1000	666 2/3
τῶν	͵Β	͵ΑΤΛΓγ′	de	2000	1333 1/3
τῶν	͵Γ	͵Β	de	3000	2000
τῶν	͵Δ	͵ΒΧΞϚ𐅷	de	4000	2666 2/3
τῶν	͵Ε	͵ΓΤΛΓγ′	de	5000	3333 1/3
τῶν	͵Ϛ	͵Δ	de	6000	4000
τῶν	͵Ζ	͵ΔΧΞϚ𐅷	de	7000	4666 2/3
τῶτ	͵Η	͵ΕΤΛΓγ′	de	8000	5333 1/3
τῶν	͵Θ	͵Ϛ	de	9000	6000
τῶν	Ι	͵ϚΧΞϚ𐅷	de 10000		6666 2/3

Fig. IV.30 The Greek text and transcription of a table of 2/3 of the whole numbers: units 1-10, tens 20-100, hundreds 200-1000, thousands 2000-10000, given in the first part of the Papyrus of Akhmin, dating from about the 7th or 8th cent. A.D. Taken from J. Baillet, *Le Papyrus mathématique d'Akhmin* (Paris, 1892), p. 24.

TABLE 4.3
Two-thirds of unit fractions.

$\overline{\overline{3}}$	of	$\overline{2}$	=	$\overline{4}$	$\overline{12}$	=	$\overline{3}$
$\overline{\overline{3}}$		$\overline{3}$		$\overline{6}$	$\overline{18}$		
$\overline{\overline{3}}$		$\overline{4}$		$\overline{8}$	$\overline{24}$		$\overline{6}$
$\overline{\overline{3}}$		$\overline{5}$		$\overline{10}$	$\overline{30}$		
$\overline{\overline{3}}$		$\overline{6}$		$\overline{12}$	$\overline{36}$		$\overline{9}$
$\overline{\overline{3}}$		$\overline{7}$		$\overline{14}$	$\overline{42}$		
$\overline{\overline{3}}$		$\overline{8}$		$\overline{16}$	$\overline{48}$		$\overline{12}$
$\overline{\overline{3}}$		$\overline{9}$		$\overline{18}$	$\overline{54}$		
$\overline{\overline{3}}$		$\overline{10}$		$\overline{20}$	$\overline{60}$		$\overline{15}$
$\overline{\overline{3}}$		$\overline{11}$		$\overline{22}$	$\overline{66}$		
$\overline{\overline{3}}$		$\overline{12}$		$\overline{24}$	$\overline{72}$		$\overline{18}$.

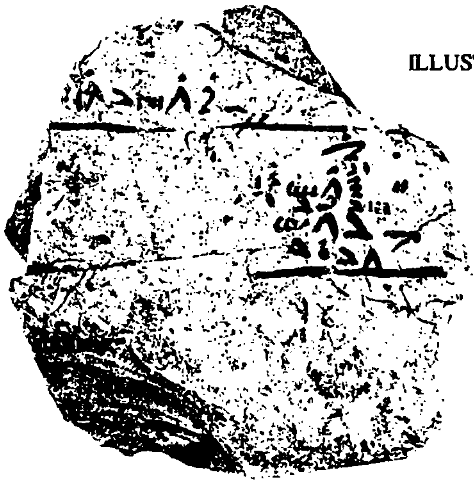
TABLE 4.4
One-third of unit fractions.

$\overline{3}$	of	$\overline{2}$	=	$\overline{8}$	$\overline{24}$	=	$\overline{6}$
$\overline{3}$		$\overline{3}$		$\overline{12}$	$\overline{36}$		$\overline{9}$
$\overline{3}$		$\overline{4}$		$\overline{16}$	$\overline{48}$		$\overline{12}$
$\overline{3}$		$\overline{5}$		$\overline{20}$	$\overline{60}$		$\overline{15}$
$\overline{3}$		$\overline{6}$		$\overline{24}$	$\overline{72}$		$\overline{18}$
$\overline{3}$		$\overline{7}$		$\overline{28}$	$\overline{84}$		$\overline{21}$
$\overline{3}$		$\overline{8}$		$\overline{32}$	$\overline{96}$		$\overline{24}$
$\overline{3}$		$\overline{9}$		$\overline{36}$	$\overline{108}$		$\overline{27}$
$\overline{3}$		$\overline{10}$		$\overline{40}$	$\overline{120}$		$\overline{30}$
$\overline{3}$		$\overline{11}$		$\overline{44}$	$\overline{132}$		$\overline{33}$
$\overline{3}$		$\overline{12}$		$\overline{48}$	$\overline{144}$		$\overline{36}$.

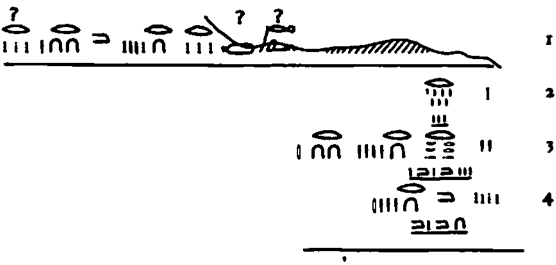
TABLE 4.5
One-half of unit fractions.

$\overline{2}$	of	$\overline{2}$	=	$\overline{4}$
$\overline{2}$		$\overline{3}$		$\overline{6}$
$\overline{2}$		$\overline{4}$		$\overline{8}$
$\overline{2}$		$\overline{5}$		$\overline{10}$
$\overline{2}$		$\overline{6}$		$\overline{12}$
$\overline{2}$		$\overline{7}$		$\overline{14}$
$\overline{2}$		$\overline{8}$		$\overline{16}$
$\overline{2}$		$\overline{9}$		$\overline{18}$
$\overline{2}$		$\overline{10}$		$\overline{20}$
$\overline{2}$		$\overline{11}$		$\overline{22}$
$\overline{2}$		$\overline{12}$		$\overline{24}$.

Fig. IV.31 Tables of two-thirds, one-third, and one-half of unit fractions as reconstructed in the Egyptian manner by R.J. Gillings, *Mathematics in the Time of the Pharaohs* (Cambridge, Mass., 1972), p. 31. Note that Gillings has used the reciprocal form to write the unit fractions.



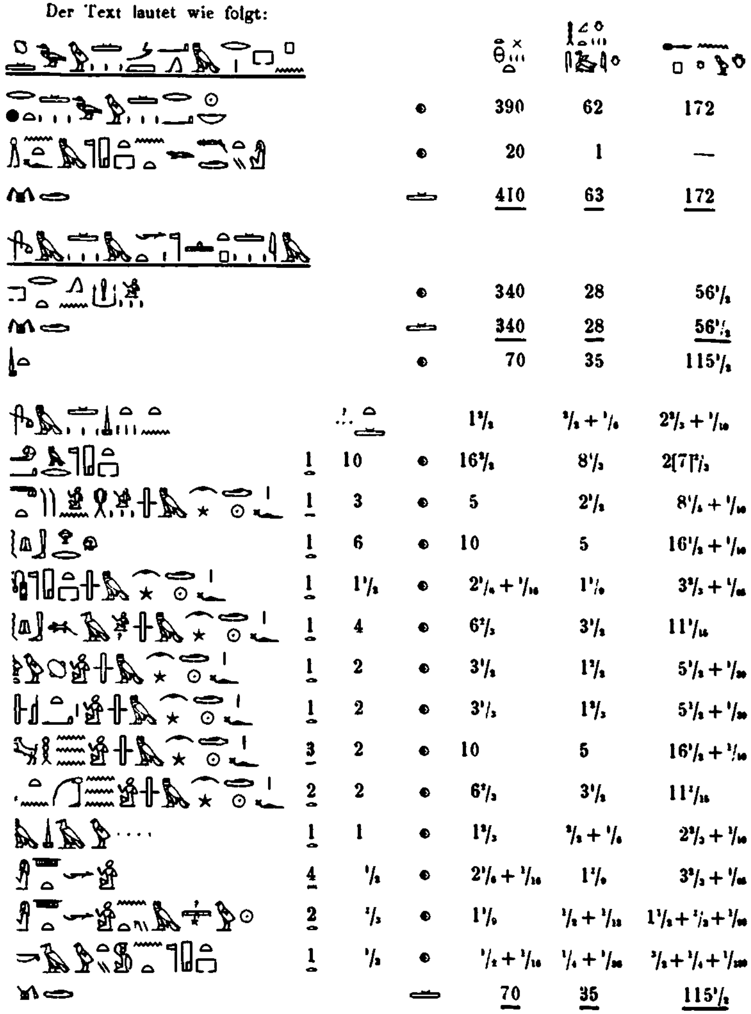
153



153

Fig. IV.32 A computation on an ostracon (no. 153) written in hieratic characters and transcribed into hieroglyphic signs. Found under the Tomb of Senmut (No. 71). Dynasty 18. The hieratic text is taken from Plate XXIX and the hieroglyphic transcription from its facing page in W.C. Hayes, *Ostraka and Name Stones from the Tomb of Sen-Mut (No. 71) at Thebes. The Metropolitan Museum of Art, Egyptian Expedition Publications*, Volume XV (New York, 1942).

Der Text lautet wie folgt:



			390	62	172
			20	1	—
			410	63	172
			340	28	56 1/2
			340	28	56 1/2
			70	35	115 1/2
			1 2/3	2/3 + 1/6	2 2/3 + 1/10
	1	10	16 2/3	8 1/3	2[7] 1/2
	1	3	5	2 1/2	8 1/4 + 1/10
	1	6	10	5	16 1/2 + 1/10
	1	1 1/2	2 1/4 + 1/16	1 1/8	3 2/3 + 1/48
	1	4	6 2/3	3 1/3	11 1/16
	1	2	3 1/3	1 2/3	5 1/2 + 1/30
	1	2	3 1/3	1 2/3	5 1/2 + 1/30
	3	2	10	5	16 1/2 + 1/10
	2	2	6 2/3	3 1/3	11 1/16
	1	1	1 2/3	2/3 + 1/6	2 2/3 + 1/10
	4	1/2	2 1/4 + 1/16	1 1/8	3 2/3 + 1/48
	2	1/3	1 1/9	1/2 + 1/18	1 1/2 + 1/3 + 1/90
	1	1/2	1/2 + 1/16	1/4 + 1/32	1/2 + 1/4 + 1/160
			70	35	115 1/2

Fig. IV.33 The hieroglyphic text of a list of Salary Portions for temple personnel at the Temple of Illahun in the Middle Kingdom. Taken from L. Borchardt, "Besoldungsverhältnisse von Priestern im mittleren Reich," ZÄS, Vol 40 (1902 03), p. 114.

		Brote	Bier	
•Berechnung der zu diesem Tempel gebrachten Einkünfte•		Stück	*Sdj*-Krüge	*Ḥpnw*-Krüge
•Liste der täglichen Einkünfte•		390	62	172
•Vom Tempel des Sobk von Krokodilopolis wurde gebracht• .		20	1	—
•Zusammen•		410	63	172
•Aufstellung(?), nachdem davon geopfert worden•				
•Ausgegeben an die Totenpriester•		340	28	$56\frac{1}{2}$
•Zusammen•		340	28	$56\frac{1}{2}$
•Rest•		70	35	$115\frac{1}{2}$
•Aufstellung (? Verteilung?) dieses Restes•	Teile(?) :	$1\frac{2}{3}$	$\frac{2}{3} + \frac{1}{6}$	$2\frac{2}{3} + \frac{1}{10}$
•Erster und Tempelvorsteher• .	1 10 :	$16\frac{2}{3}$	$8\frac{1}{3}$	$2[7]\frac{1}{2}$
•Vorsteher der Laienpriesterabteilung, der in diesem Monat Dienst hat•	1 3 :	5	$2\frac{1}{2}$	$8\frac{1}{6} + \frac{1}{10}$
•Hauptvorlesepriester•	1 6 :	10	5	$16\frac{1}{3} + \frac{1}{10}$
•Tempelschreiber, der in diesem Monat Dienst hat•	1 $1\frac{1}{2}$:	$2\frac{1}{4} + \frac{1}{18}$	$1\frac{1}{6}$	$3\frac{2}{3} + \frac{1}{43}$
•Gewöhnlicher Vorlesepriester, der in diesem Monat Dienst hat• :	1 4 :	$6\frac{2}{3}$	$3\frac{1}{3}$	$11\frac{1}{16}$
•*Wb*-Priester, der in diesem Monat Dienst hat•	1 2 :	$3\frac{1}{3}$	$1\frac{2}{3}$	$5\frac{1}{3} + \frac{1}{30}$
•*Imi-ist-ʿ*-Priester, der in diesem Monat Dienst hat	1 2 :	$3\frac{1}{3}$	$1\frac{2}{3}$	$5\frac{1}{3} + \frac{1}{30}$
•*Ibḥ*-Priester, der in diesem Monat Dienst hat•	3 2 :	10	5	$16\frac{1}{3} + \frac{1}{10}$
•Königlicher Priester, der in diesem Monat Dienst hat• . .	2 2 :	$6\frac{2}{3}$	$3\frac{1}{3}$	$11\frac{1}{16}$
•*Mdw*•	1 1 :	$1\frac{2}{3}$	$\frac{2}{3} + \frac{1}{6}$	$2\frac{2}{3} + \frac{1}{30}$
•Thürhüter•	4 $\frac{1}{3}$:	$2\frac{1}{6} + \frac{1}{18}$	$1\frac{1}{9}$	$3\frac{2}{3} + \frac{1}{45}$
•Thürhüter, der nachts Dienst hat•	2 $\frac{1}{2}$:	$1\frac{1}{6}$	$\frac{1}{3} + \frac{1}{18}$	$1\frac{1}{3} + \frac{1}{3} + \frac{1}{90}$
•Tempelarbeiter•	1 $\frac{1}{3}$:	$\frac{1}{2} + \frac{1}{18}$	$\frac{1}{4} + \frac{1}{36}$	$\frac{2}{3} + \frac{1}{4} + \frac{1}{180}$
•Zusammen•		70	35	$115\frac{1}{2}$

Fig. IV.34 A German translation of Fig. IV.33 made by Borchardt, *ibid.*, p. 115.

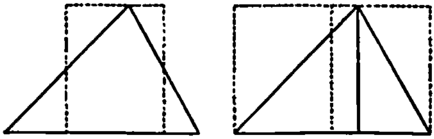


Fig. IV.35a Suggested graphic solution of a scalene triangle. Taken from T.E. Peet, "Mathematics in Ancient Egypt," p. 432.

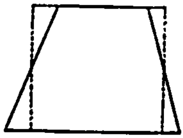


Fig. IV.35b Suggested graphic solution of a truncated triangle, i.e a trapezoid. Taken from Peet, *Ibid.*, p. 433.

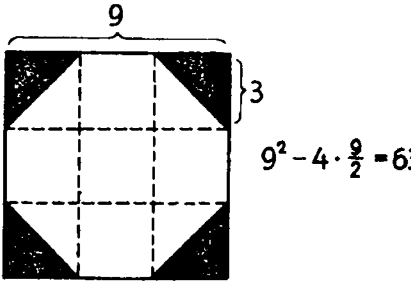


Fig. IV.36 The octagon of Problem 48 of Document IV.1, redrawn as a symmetrical octagon inscribed in a square by K. Vogel, *Vorgriechische Mathematik*, Teil I, p. 66.

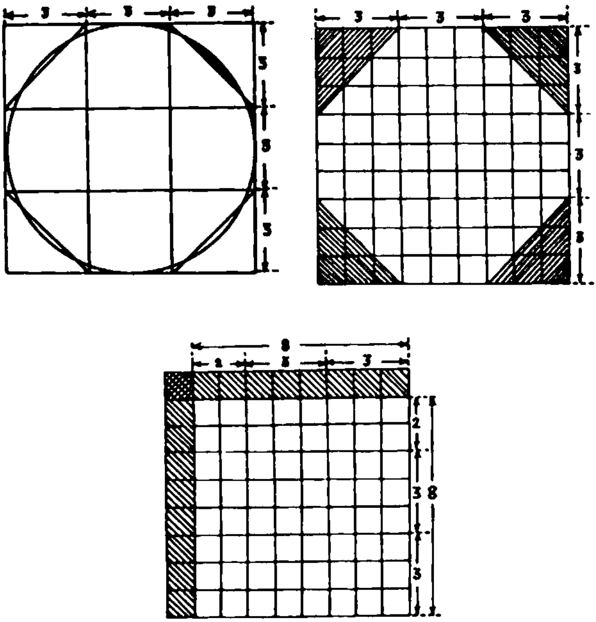


Fig. IV.37 Three figures illustrating Gillings' suggested graphic solution of the Egyptian formula for the area of a circle. Taken from Gillings, *Mathematics in the Time of the Pharaohs*, p. 144.

<table>
<tr><th>Text</th><th>Struve</th><th>Peet</th></tr>
<tr><td>Beispiel zur Berechnung einer nb.t
Wenn man Dir sagt: eine nb.t
m tp-r3
r $4\frac{1}{2}$ m 'd̲
laß mich wissen ihre Fläche
Nimm $\bar{9}$ von 9
weil die nb.t die Hälfte des i ▮▮ ist
das macht 1</td><td>Korb = Halbkugel

von Mündung (d)
zu $4\frac{1}{2}$ in Erhaltung

i[nr] = Kugel</td><td>Korb = Halbzylinder

[von $4\frac{1}{2}$] an Mündung (d)
zu $4\frac{1}{2}$ an 'd̲ (a)

i[p.t]</td></tr>
<tr><td>9 − 1 = 8</td><td colspan="2">$2d - \frac{1}{9}2d$</td></tr>
<tr><td>$\bar{9} \cdot 8 = 3 + 6 + \overline{18}$
$8 - (3 + 6 + \overline{18}) = 7 + \bar{9}$</td><td colspan="2">$\left(2d - \frac{1}{9}2d\right) - \frac{1}{9}\left(2d - \frac{1}{9}2d\right) = \left(\frac{8}{9}\right)^2 2d \approx \frac{\pi}{4}2d$</td></tr>
<tr><td>$(7 + \bar{9})(4 + \bar{2}) = 32$ = Fläche</td><td>$F = d\left\{(\ \ ) - \frac{1}{9}(\ \ )\right\} \approx \frac{d^2\pi}{2}$</td><td>$F = a\left\{(\ \ ) - \frac{1}{9}(\ \ )\right\} \approx a\frac{d\pi}{2}$</td></tr>
</table>

Fig. IV.38 Comparison of the interpretations of Problem 10 of the Moscow Mathematical Papyrus by Struve and Peet. Given by Neugebauer, *Vorgriechische Mathematik*, p. 130.

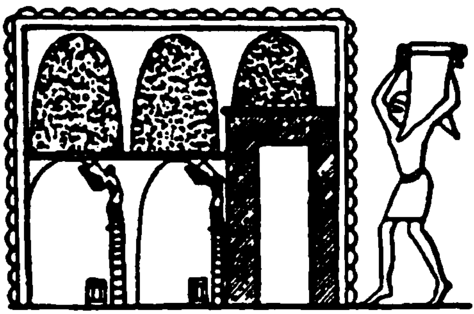
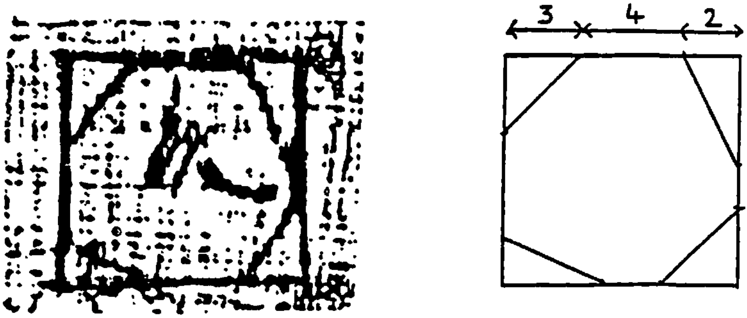


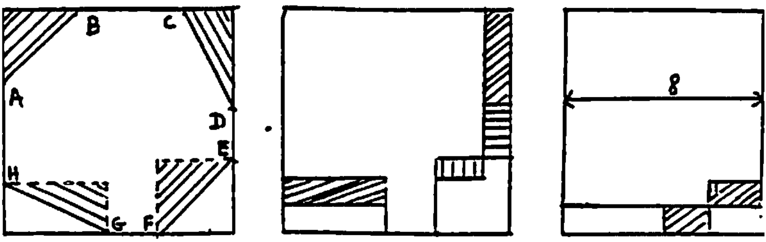
Fig. IV.39 An illustration of Egyptian dome-like granaries used in Neugebauer's suggested interpretation of Problem 10 of the Moscow Mathematical Papyrus, *ibid.*, p. 136.



Nous avons bien pour l'aire de l'octogone ainsi construit :

$9^2 - (3^2 + 2.4) = 81 - (9 + 8) = 81 - 17 = 64.$

Connaissant les mesures des aires des triangles rectangles, le scribe pouvait ainsi calculer la mesure de l'aire de l'octogone. Mieux, même, compte tenu de la symétrie de la figure il pouvait "paver" l'octogone selon le schéma suivant :



Autrement dit, "géométriquement", l'aire de l'octogone ABCDEFGH est égale à l'aire du carré PQRS. Certes il ne s'agit que d'une reconstruction, mais d'une reconstruction tout à fait plausible compte tenu des connaissances des anciens Egyptiens. Mais le problème ne réside pas essentiellement dans l'existence d'une telle reconstiution ; il se situe au contraire dans la considération de l'octogone ABCDEFGH. Ce sera principalement l'objet de notre conclusion.

Fig. IV.40 The figure in the Rhind Papyrus for Problem 48 and its redrawing as suggested by M. Guillemot, 'A propos de la 'géométrie égyptienne des figures',' p. 139. Included is his computation of the area of the octagon as 64, i.e., as equal to a square of side 8, which the Egyptians assumed as approximately equivalent to a circle of diameter 9. The reference to square PQRS in the bottom paragraph is to the square of side 8 in the bottom right figure.

Arithmétiquement, dans le cadre égyptien des quantièmes, nous avons à rechercher l'entier naturel n tel que $\frac{\Pi}{4}$ soit le plus voisin possible $\left(1 - \frac{1}{n}\right)^2$. Or nous avons

$$\frac{\Pi}{4} \approx 0{,}785 \quad \left(1 - \frac{1}{8}\right)^2 \approx 0{,}766 \quad \left(1 - \frac{1}{9}\right)^2 \approx 0{,}790 \quad \left(1 - \frac{1}{10}\right)^2 = 0{,}81.$$

Nous constatons que 9 est l'entier qui convient le mieux et c'est précisément celui choisi par les anciens Egyptiens. Nous pouvons noter que :

$$\left(\frac{8}{9} - \frac{1}{375}\right)^2 \approx 0{,}785390 < \frac{\Pi}{4} \approx 0{,}785398 < \left(\frac{8}{9} - \frac{1}{376}\right)^2 \approx 0{,}785402.$$

Autrement dit, l'amélioration de l'approximation est plus délicate à mettre en oeuvre surtout si nous tenons compte des conditions qui président à certains calculs. Il n'en demeure pas moins que l'approximation choisie par les anciens Egyptiens est la meilleure dans le cadre d'un seul quantième.

Nous savons que la division ou la multiplication par deux était la méthode opératoire préférée des anciens Egyptiens. Dans ce cadre il eût été plus facile de choisir $\left(1 - \frac{1}{8}\right)^2$ à la place de $\left(1 - \frac{1}{9}\right)^2$. Nous pouvons donc considérer que l'approximation a été conduite avec un soin certain. Néanmoins l'introduction du neuvième n'offre pas de difficulté insurmontable compte tenu des relations suivantes :

$$\frac{2}{3}D = \frac{1}{2}D + \frac{1}{6}D. \qquad \frac{1}{3}D = \frac{1}{4}D + \frac{1}{12}D$$

$$\frac{1}{9}D = \frac{1}{12}D + \frac{1}{36}D$$

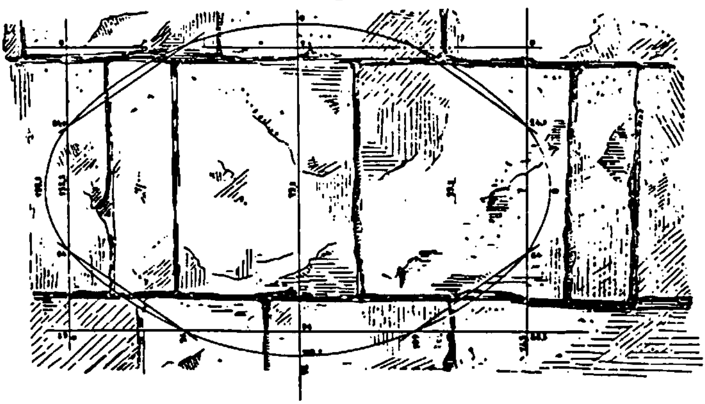
$$D - \frac{1}{9}D = \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{12} + \frac{1}{18} + \frac{1}{36}\right)D - \left(\frac{1}{12} + \frac{1}{36}\right)D = \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{18}\right)D$$

$$= \left(\frac{2}{3} + \frac{1}{6} + \frac{1}{18}\right)D. \text{ (voir R42).}$$

Mais cette excellente approximation arithmétique nous pousse à rejeter une heuristique "géométrique". Autrement dit, "l'exemple" 48 doit être considéré comme étant une "explication géométrique" de la formule mise en oeuvre. La figure et les calculs afférents jouent le même rôle que les signes des écritures égyptiennes. Ce n'est sans doute pas un hasard si certains problèmes sont réduits à cette seule présentation ou si celle-ci constitue la deuxième partie de R51. Les commentaires ne servent alors qu'à dévoiler les explications plus ou moins secrètes qui sont absentes d'un tel schéma. Soit par incapacité, soit par un souci de sauvegarde, le scribe n'a pas voulu tout nous dire lorsqu'il a écrit le "problème" R48. Mais ce dernier représente un moment important dans l'histoire de la pensée mathématique : celui d'un essai de "justification géométrique" d'un résultat obtenu empiriquement.

Fig. IV.41 M. Guillemot's remarks on the computation of the quadrature rule for the area of a circle, as given in his article quoted in the legend of Fig. IV 40, p. 140. R.42, R.48, R.51, and R.52 refer to the numbered problems of the Rhind Papyrus.

Fig. VII.



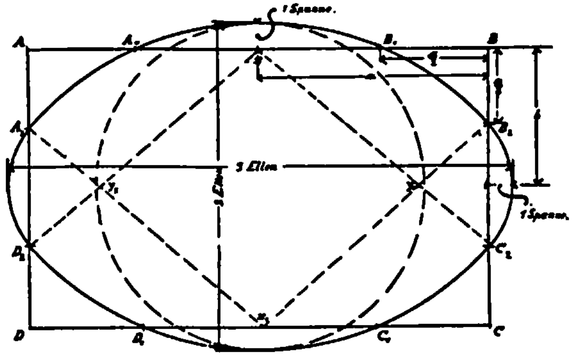


Fig. IV.42 An ellipse and an approximately equal rectangle scratched on a wall in the Temple of Luxor, with Burchardt's interpretative drawing below. Taken from L. Borchardt, "Altägyptische Werkzeichnung," *ZÄS*, Vol. 34 (1896), Pl. VI, Fig. 7.

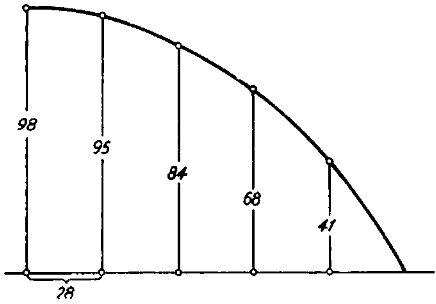


Fig. IV.43 A portable Sketch of an arc on limestone from Dynasty 3 at Saqqara, showing the use of ordinates of specified lengths in cubits, palms, and fingers to guide the form of the arc. The ordinates are placed 1 cubit apart and marked as follows: 3 cub. + 3 palms + 2 fing. [equaling 98 fing.], 3 cub. + 2 palms + 3 fing. [equaling 95 fing.], 3 cub. [equaling 84 fing.], 2 cub. + 3 palms [equaling 68 fing.], and 1 cub. + 3 palms + 1 fing. [equaling 41 fing.]. The figure below is the suggested completion of the arc and its ordinates, with the lengths translated into modern numerals and with the measures converted into fingers on the basis of the royal cubit equal to 28 fingers and the palm to 4 fingers. Both figures are taken from G. Wolff, "Ägyptische Mathematik in Kunst und Handwerk," pp. 266-67.